Towards Understanding Normalization in Deep Learning and its Applications

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The Chinese University of Hong Kong (CUHK)

VALSE Webinar, 2018-10-24

Outline

1. Whitened Neural Network (WNN) *(Desjardins etal. NIPS15)*
   - Relations between network design and optimization
   - Post-GWNN *(Luo ICML17)*
2. Batch Normalization (BN)
   - Regularization and Generalization *(Luo etal. arXiv:1809.00846)*
3. Switchable Normalization (SN) *(Luo etal. arXiv:1806.10779)*
4. More Techniques
   - Kalman Normalization (KN) *(Wang etal. NIPS18)*
   - Instance-Batch Normalization Network (IBN-Net) *(Pan etal. ECCV18)*
5. Discussions and Future Work

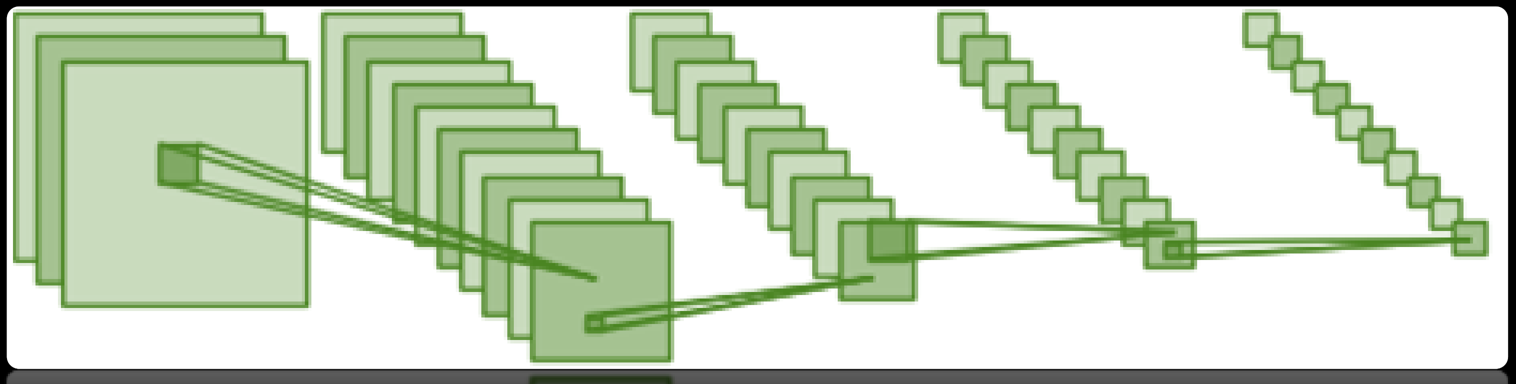
Whitened Neural Network (WNN)

PART 1

Relations between Network Design and Optimization

$$y = f(x;\theta)$$

Input: x



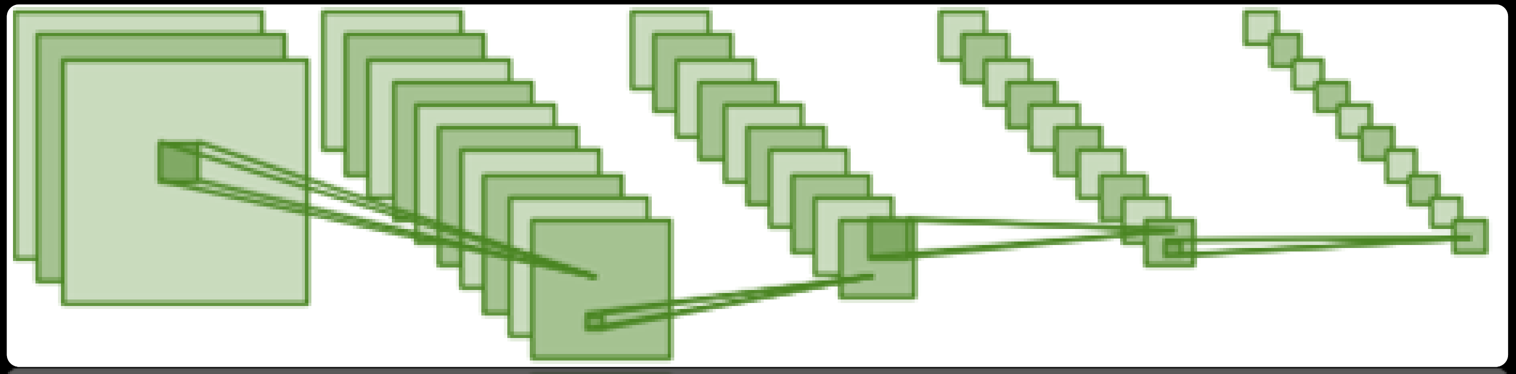
Label: $\hat{y}$

$$\theta^{t+1} = \theta^{t} - F^{-1}\nabla_{\theta}\mathcal{L}(\hat{y}, f(x;\theta))$$

Relations between Network Design and Optimization

$$\boxed{y = f(x;\theta)}$$

Input: x



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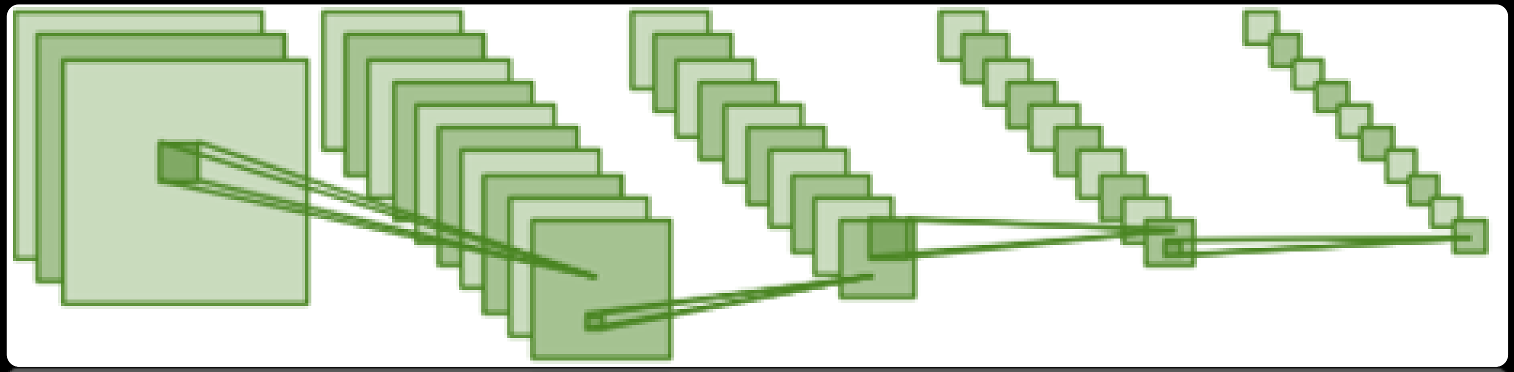

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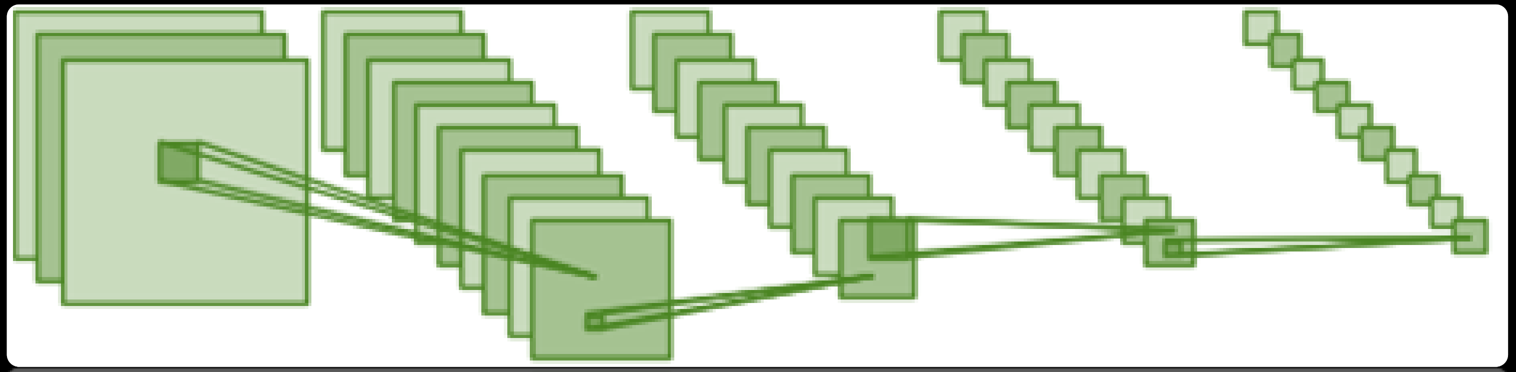


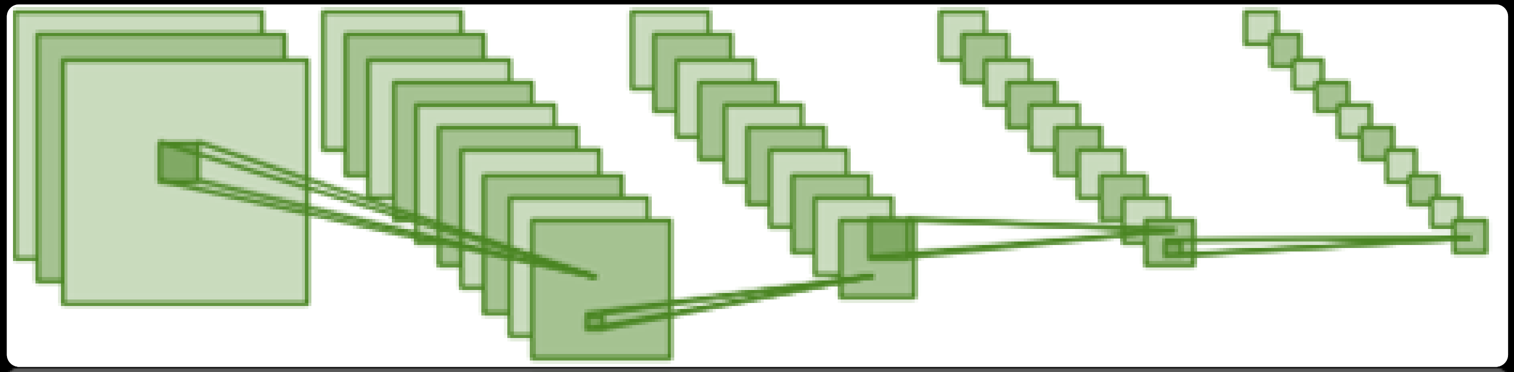
Label: $\hat{y}$

Natural Gradient Descent (NGD): $\theta^{t+1} = \theta^{t} - F^{-1}\boxed{\nabla_\theta \mathcal{L}(\hat{y}, f(x;\theta))}$

Relations between Network Design and Optimization

$$y = f(x;\theta)$$

Input: x



Label: $\hat{y}$

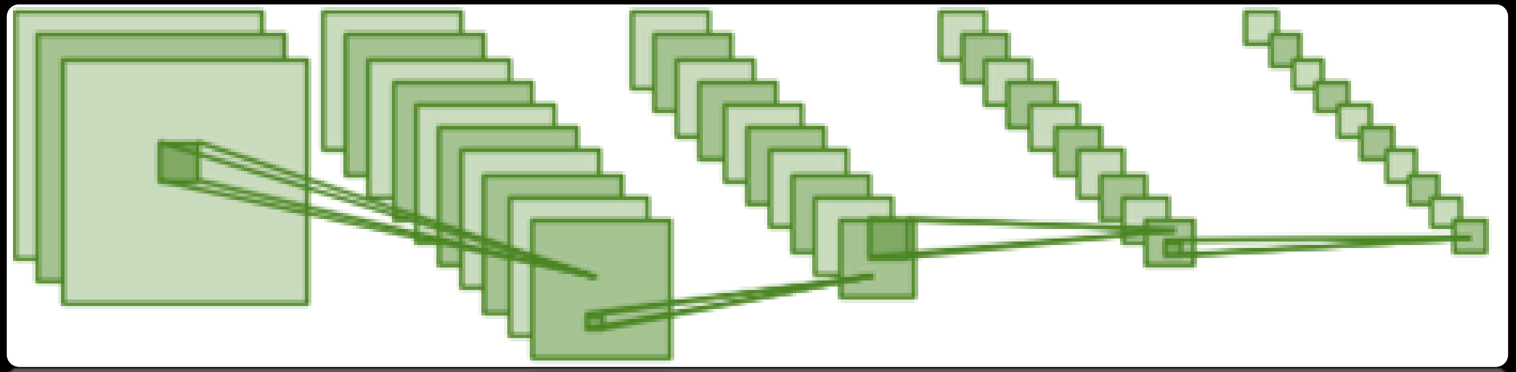
Natural Gradient Descent (NGD): $\theta^{t+1} = \theta^{t} - \boxed{F^{-1}} \nabla_\theta \mathcal{L}(\hat{y}, f(x;\theta))$

F: Fisher Information Matrix (FIM)

Relations between Network Design and Optimization

$$y = f(x;\theta)$$

Input: x



Label: $\hat{y}$

Natural Gradient Descent (NGD): $\theta^{t+1} = \theta^{t} - F^{-1}\nabla_{\theta}\mathcal{L}(\hat{y}, f(x;\theta))$

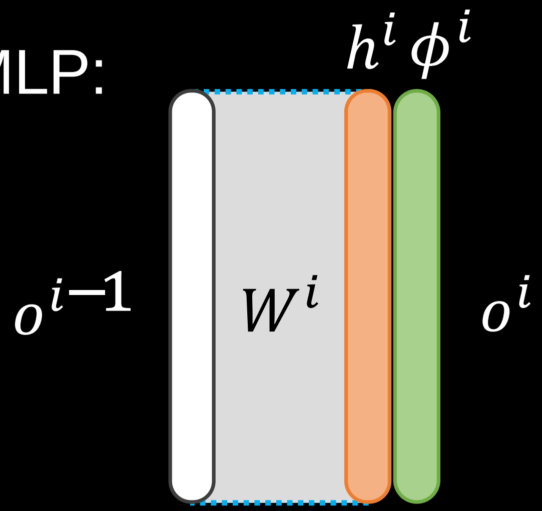
$\boxed{\text{SGD: } F = I}$ works well.

Relations between Network Design and Optimization

Deep Learning turns Optimization Problems into Feed-Forward Network Design.

Multilayer Perceptron (MLP)

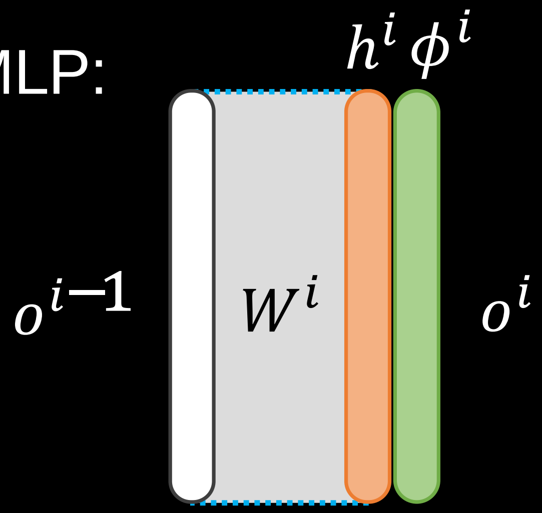
MLP:



Forward：

Multilayer Perceptron (MLP)

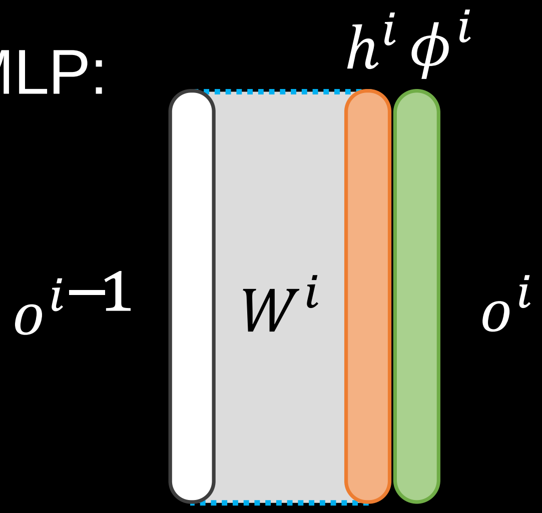
MLP:



Forward: $h^i = W^i o^{i-1}$ (linear transformation)

Multilayer Perceptron (MLP)

MLP:



layer input

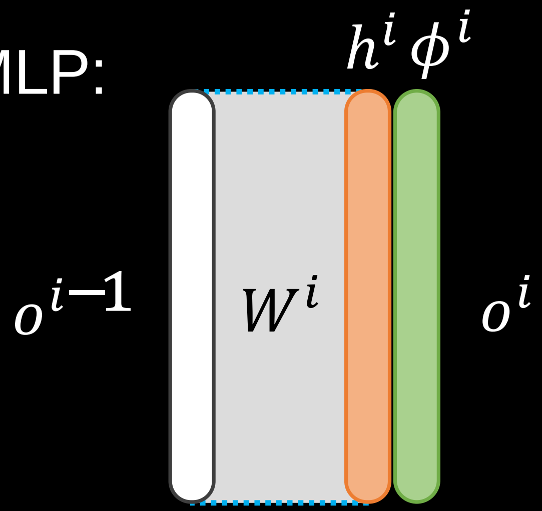
Forward: $h^i = W^i o^{i-1}$ (linear transformation)

hidden feature

network parameter

Multilayer Perceptron (MLP)

MLP:

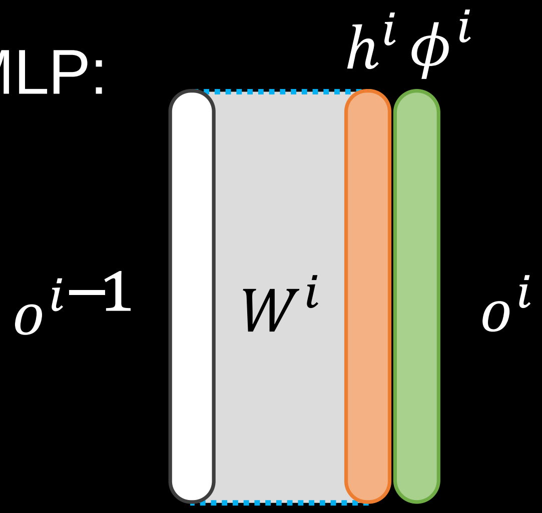


Forward: $h^i = W^i o^{i-1}$ (linear transformation)

$$\phi^i = \frac{h^i - \mathrm{E}[h^i]}{\sqrt{\mathrm{Var}(h^i)}}$$ (batch normalization)

Multilayer Perceptron (MLP)

MLP:



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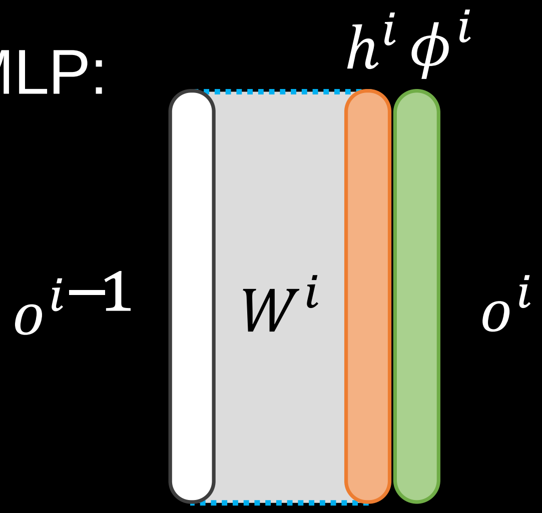
Multilayer Perceptron (MLP)

MLP:



Forward： $h^i = W^i o^{i-1}$ (linear transformation)

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- Natural gradient descent (NGD) with Fisher information matrix (FIM) F, $W_{t+1}^i = W_t^i - \lambda_t F_t^{-1} \nabla W_t^i$.

WNN is NGD

A DNN with l layers.

$$F = \left[\begin{array}{c:c:c:c} F_{11} & & & F_{ij} \\ \hdashline & F_{22} & & \\ \hdashline & & \ddots & \\ \hdashline F_{ij} & & & F_{ll} \end{array}\right]$$

- Natural gradient with Fisher information matrix (FIM) F is defined as $W_{t+1}^{i} = W_{t}^{i} - \lambda_t F_t^{-1} \nabla W_t^i$.

WNN is NGD

A DNN with l layers.

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- Natural gradient with Fisher information matrix (FIM) F is defined as $W_{t+1}^{i} = W_t^i - \lambda_t F_t^{-1} \nabla W_t^i$.

WNN is NGD

Covariance of gradients

$$F_{ij} = \mathbb{E}[\mathrm{vec}(\nabla W^i)\mathrm{vec}(\nabla W^j)^T]$$

A DNN with l layers.

$$F = \left[\begin{array}{c:c:c:c} F_{11} & & & F_{ij} \\ \hdashline & F_{22} & & \\ \hdashline & & \ddots & \\ \hdashline F_{ij} & & & F_{ll} \end{array}\right]$$

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WNN is NGD

vectorization   gradient: $\nabla W^{i} = o^{i-1} \Delta h^{i^{T}}$

$$F_{ij} = \mathbb{E}[\mathrm{vec}(\nabla W^{i})\mathrm{vec}(\nabla W^{j})^{T}]$$

A DNN with l layers.

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WNN is NGD

vectorization gradient: $\nabla W^i = o^{i-1} {\Delta h^i}^T$

$$F_{ij} = \mathbb{E}[\mathrm{vec}(\nabla W^i)\mathrm{vec}(\nabla W^j)^T]$$

$$= \mathbb{E}[(\Delta h^i \otimes o^{i-1})(\Delta h^j \otimes o^{j-1})]$$

A DNN with l layers.

$$F = \begin{bmatrix} F_{11} & & & F_{ij} \\ & F_{22} & & \\ & & \ddots & \\ F_{ij} & & & F_{ll} \end{bmatrix}$$

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WNN is NGD

vectorization gradient: $\nabla W^i = o^{i-1} {\Delta h^i}^T$

Kronecker product

$$F_{ij} = \mathbb{E}[\mathrm{vec}(\nabla W^i)\mathrm{vec}(\nabla W^j)^T]$$
$$= \mathbb{E}[(\Delta h^i \otimes o^{i-1})(\Delta h^j \otimes o^{j-1})]$$

A DNN with l layers.

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$$F_{ij} = \mathbb{E}[\mathrm{vec}(\nabla W^i)\mathrm{vec}(\nabla W^j)^T]$$

$$= \mathbb{E}[(\Delta h^i \otimes o^{i-1})(\Delta h^j \otimes o^{j-1})]$$

(mixed-product property)

$$= \mathbb{E}[\Delta h^i (\Delta h^j)^T \otimes o^{i-1}(o^{j-1})^T]$$

- Natural gradient with Fisher information matrix (FIM) F is defined as $W_{t+1}^i = W_t^i - \lambda_t F_t^{-1} \nabla W_t^i$.

WNN is NGD

A DNN with l layers.

$$F = \begin{bmatrix} F_{11} & & & F_{ij} \\ & F_{22} & & \\ & & \ddots & \\ F_{ij} & & & F_{ll} \end{bmatrix}$$

vectorization

gradient: $\nabla W^i = o^{i-1} {\Delta h^i}^T$

Kronecker product

$$F_{ij} = \mathbb{E}[\mathrm{vec}(\nabla W^i)\mathrm{vec}(\nabla W^j)^T]$$

$$= \mathbb{E}[(\Delta h^i \otimes o^{i-1})(\Delta h^j \otimes o^{j-1})]$$

(mixed-product property) $$= \mathbb{E}[\Delta h^i (\Delta h^j)^T \otimes o^{i-1}(o^{j-1})^T]$$

(independence) $$\approx \mathbb{E}[\Delta h^i (\Delta h^j)^T] \otimes \mathbb{E}[o^{i-1}(o^{j-1})^T]$$

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A DNN with l layers.

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vectorization gradient: $\nabla W^i = o^{i-1} {\Delta h^i}^T$

Kronecker product

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(independence) $\approx \boxed{\mathbb{E}[\Delta h^i (\Delta h^j)^T]} \otimes \boxed{\mathbb{E}[o^{i-1}(o^{j-1})^T]}$

- Natural gradient with Fisher information matrix (FIM) F is defined as $W^i_{t+1} = W^i_t - \lambda_t F_t^{-1} \nabla W^i_t$.

WNN is NGD

A DNN with l layers.

$$F = \left[\begin{array}{c:c:c:c} F_{11} & & & F_{ij} \\ \hdashline & F_{22} & & \\ \hdashline & & \ddots & \\ \hdashline F_{ij} & & & F_{ll} \end{array}\right]$$

vectorization

gradient: $\nabla W^i = o^{i-1} {\Delta h^i}^T$

Kronecker product

$$F_{ij} = \mathbb{E}[\mathrm{vec}(\nabla W^i)\mathrm{vec}(\nabla W^j)^T]$$

$$= \mathbb{E}[(\Delta h^i \otimes o^{i-1})(\Delta h^j \otimes o^{j-1})]$$

(mixed-product property) $= \mathbb{E}[\Delta h^i(\Delta h^j)^T \otimes o^{i-1}(o^{j-1})^T]$

(independence) $\approx \mathbb{E}[\Delta h^i(\Delta h^j)^T] \otimes \mathbb{E}[o^{i-1}(o^{j-1})^T]$

- Goal: $F_{ii} \approx I$.

WNN is NGD

A DNN with l layers. $\approx I$

$$F = \begin{bmatrix} F_{11} & & & F_{ij} \\ & F_{22} & & \\ & & \ddots & \\ F_{ij} & & & F_{ll} \end{bmatrix}$$

$$\begin{aligned} F_{ij} &= \mathbb{E}[\mathrm{vec}(\nabla W^{i})\mathrm{vec}(\nabla W^{j})^{T}] \\ &= \mathbb{E}[(\Delta h^{i} \otimes o^{i-1})(\Delta h^{j} \otimes o^{j-1})] \\ &= \mathbb{E}[\Delta h^{i}(\Delta h^{j})^{T} \otimes o^{i-1}(o^{j-1})^{T}] \\ &\approx \mathbb{E}[\Delta h^{i}(\Delta h^{j})^{T}] \otimes \mathbb{E}[o^{i-1}(o^{j-1})^{T}] \end{aligned}$$

$$\boxed{F_{ii} = \mathbb{E}[\Delta h^{i}(\Delta h^{i})^{T}] \otimes \mathbb{E}[o^{i-1}(o^{i-1})^{T}]}$$

WNN is NGD

A DNN with l layers.

$$F = \begin{bmatrix} F_{11} & & & F_{ij} \\ & F_{22} & & \\ & & \ddots & \\ F_{ij} & & & F_{ll} \end{bmatrix}$$

$$\begin{aligned} F_{ij} &= \mathbb{E}[\mathrm{vec}(\nabla W^{i})\mathrm{vec}(\nabla W^{j})^{T}] \\ &= \mathbb{E}[(\Delta h^{i} \otimes o^{i-1})(\Delta h^{j} \otimes o^{j-1})] \\ &= \mathbb{E}[\Delta h^{i}(\Delta h^{j})^{T} \otimes o^{i-1}(o^{j-1})^{T}] \\ &\approx \mathbb{E}[\Delta h^{i}(\Delta h^{j})^{T}] \otimes \mathbb{E}[o^{i-1}(o^{j-1})^{T}] \end{aligned}$$

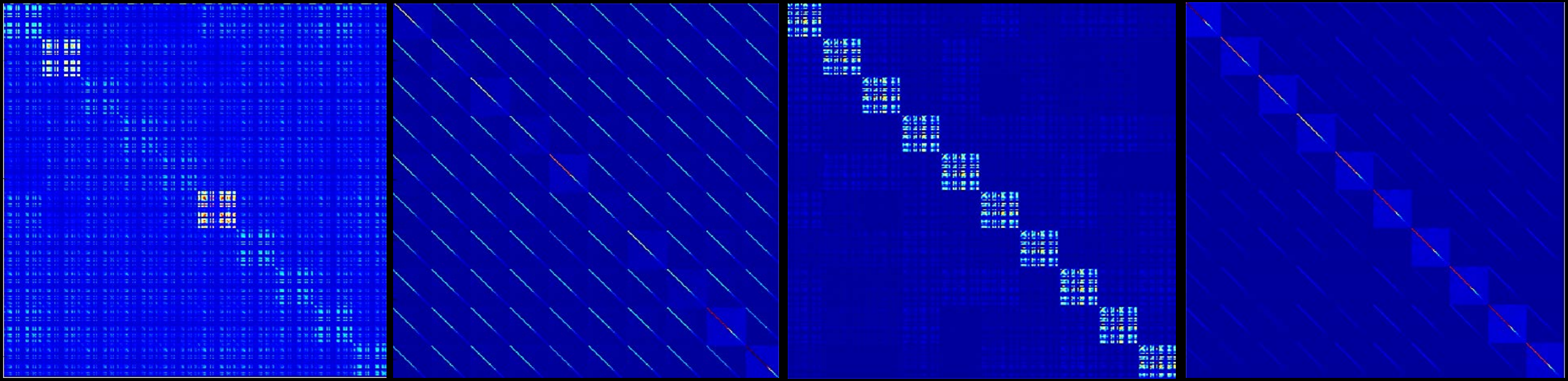
$$F_{ii} = \boxed{\mathbb{E}[\Delta h^{i}(\Delta h^{i})^{T}]} \otimes \boxed{\mathbb{E}[o^{i-1}(o^{i-1})^{T}]}$$

EigenNet, IJCAI17

WNN, NIPS15
GWNN, ICML17

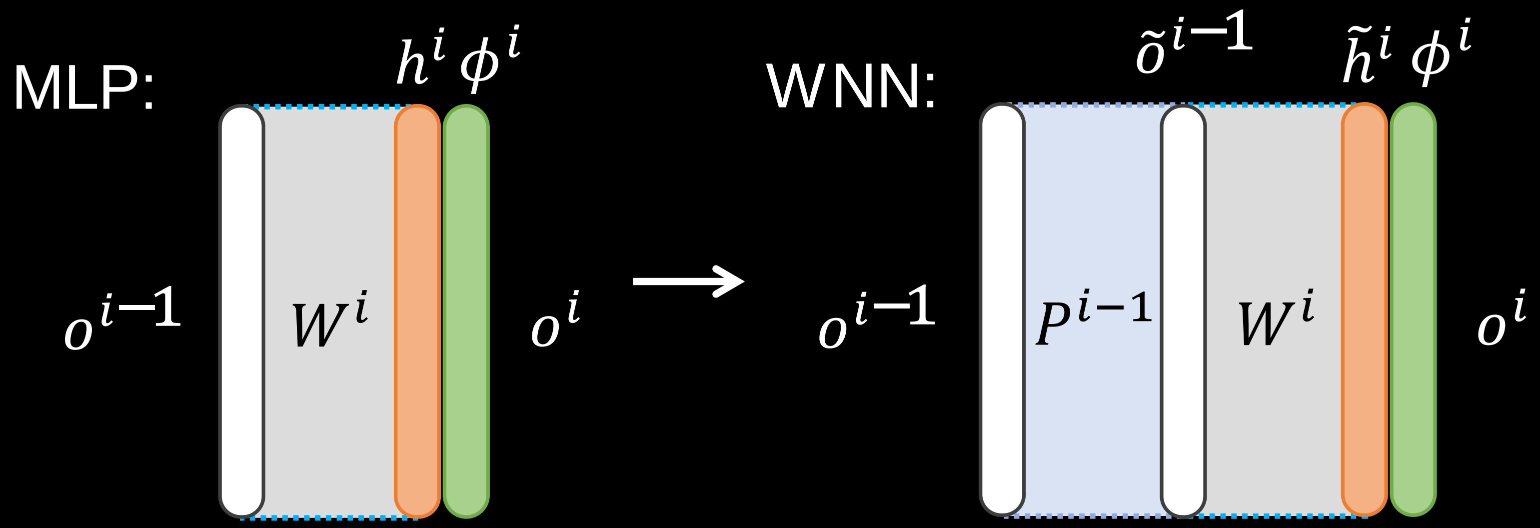
WNN is NGD

1) BN 2) Whiten forward 3) Whiten backward 4) Whiten both

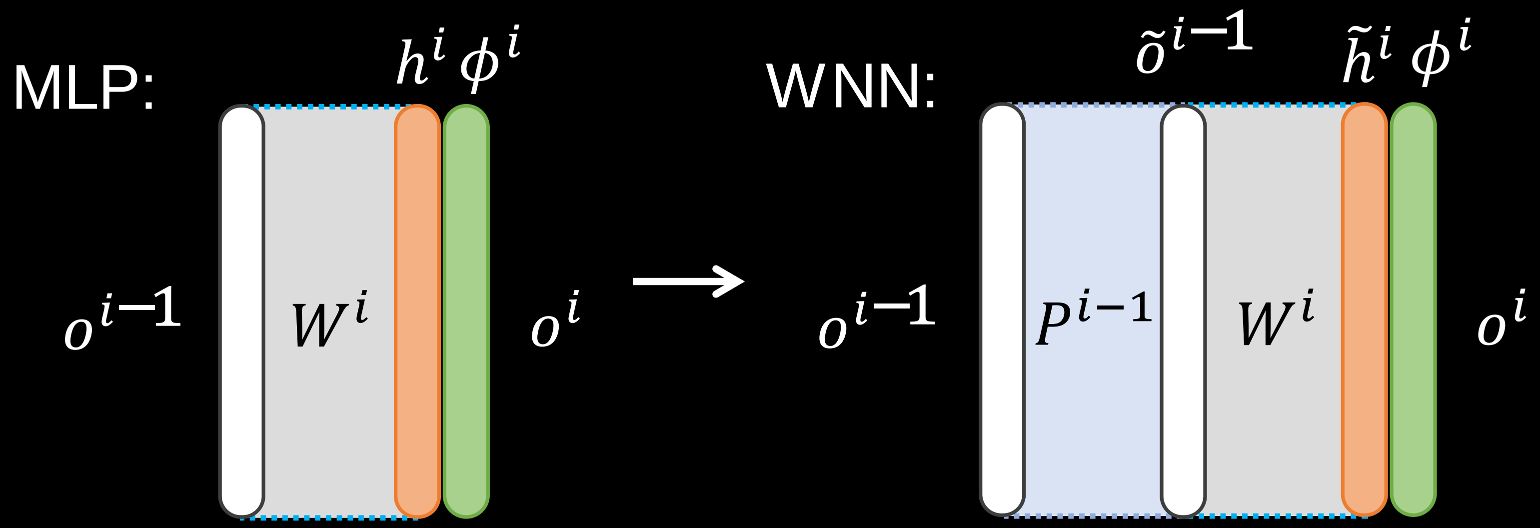


- Approximate $F_{ii} = \mathbb{E}[\Delta h^i(\Delta h^i)^T] \otimes \mathbb{E}[o^{i-1}(o^{i-1})^T] \approx I.$

Whitened Neural Network (WNN)

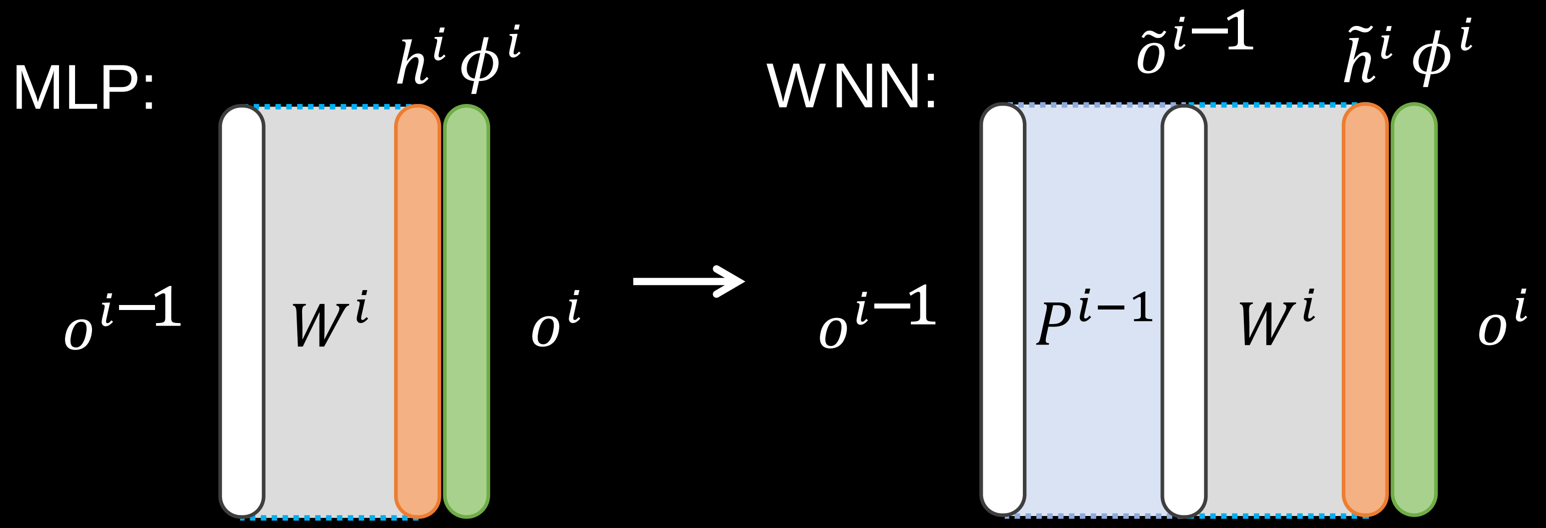


Whitened Neural Network (WNN)



$$\boxed{\min_{\{W^i\}} L}$$

Whitened Neural Network (WNN)

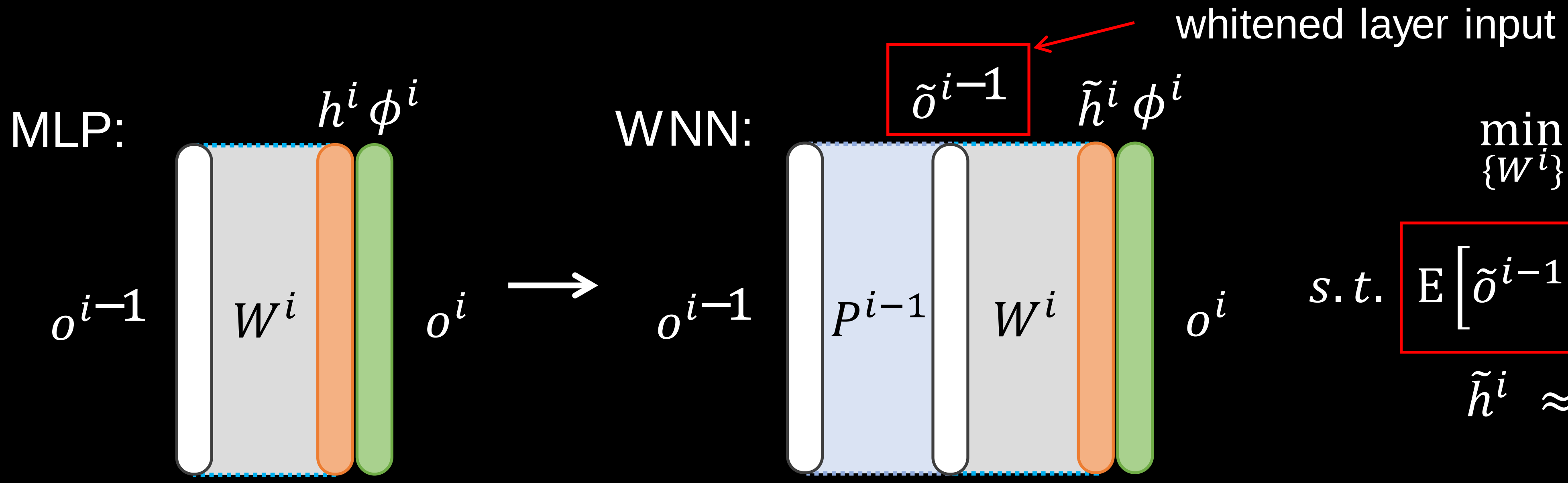


$$\min_{\{W^i\}} L$$

$$s.t.\ \mathrm{E}\left[\tilde{o}^{i-1}{\tilde{o}^{i-1}}^T\right] = I,$$

$$\tilde{h}^i \approx h^i.$$

Whitened Neural Network (WNN)

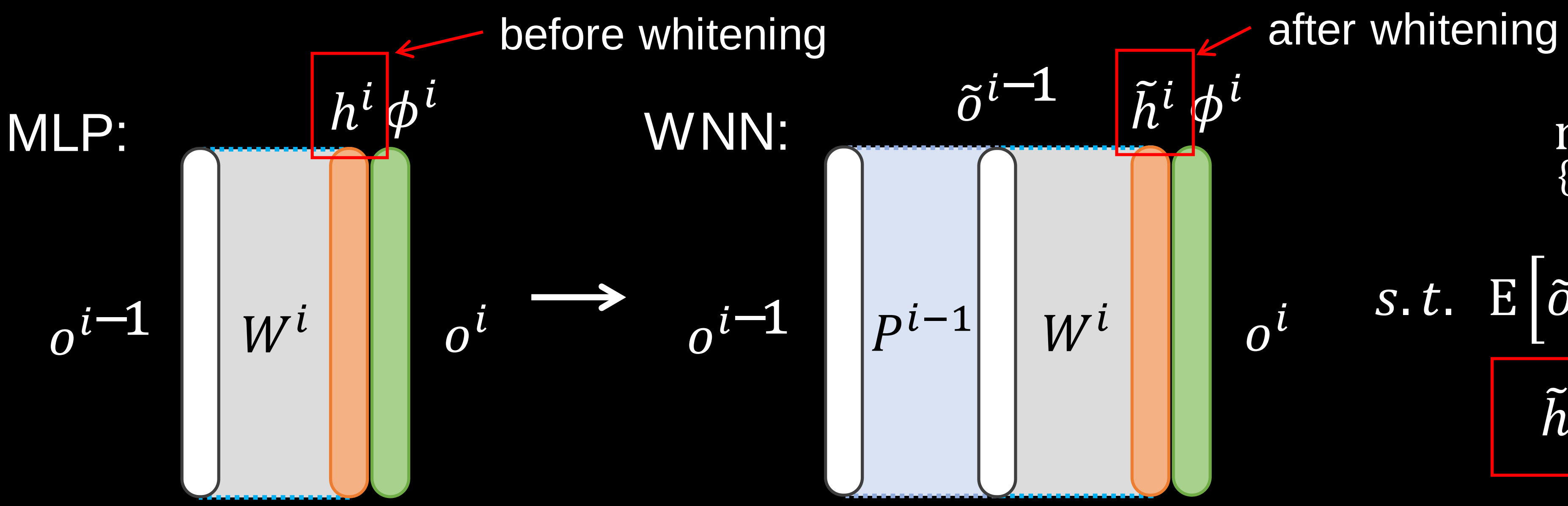


$$\min_{\{W^i\}} L$$

$$s.t.\ \boxed{\mathrm{E}\left[\tilde{o}^{i-1}{\tilde{o}^{i-1}}^{T}\right] = I,}$$

$$\tilde{h}^i \approx h^i\,.$$

Whitened Neural Network (WNN)



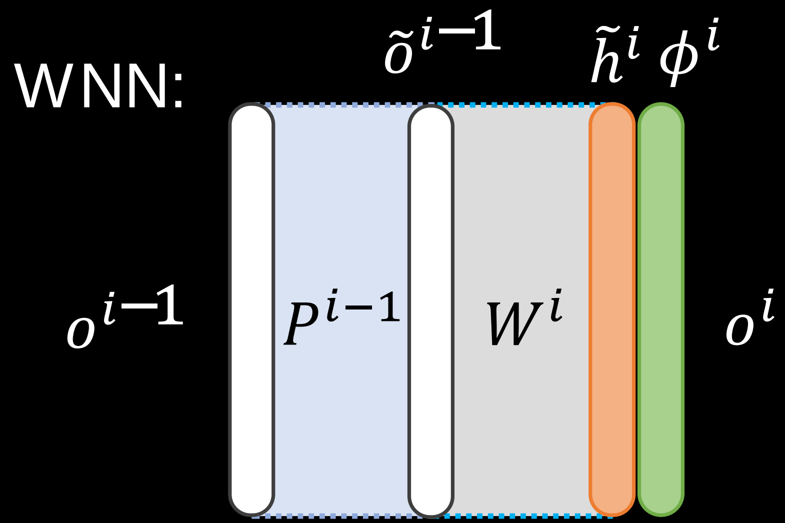
$$\min_{\{W^i\}} L$$

$$s.t.\ \ \mathrm{E}\left[\tilde{o}^{i-1}\tilde{o}^{i-1^T}\right] = I,$$

$$\boxed{\tilde{h}^i \approx h^i\,.}$$

WNN smooth solution space while maintains representation capacity.

Whitened Neural Networks (WNN)

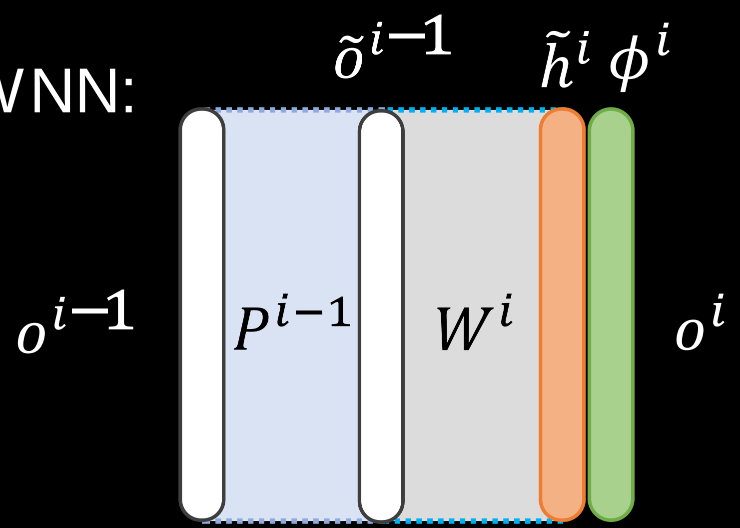


Forward: $\tilde{o}^{i-1} = P^{i-1}(o^{i-1} - \mu^{i-1}), \mathrm{E}\left[\tilde{o}^{i-1}{\tilde{o}^{i-1}}^T\right] = I$

(whitening)

Whitened Neural Networks (WNN)

WNN:



Forward: $\tilde{o}^{i-1} = P^{i-1}(o^{i-1} - \mu^{i-1}), \mathrm{E}\left[\tilde{o}^{i-1}\tilde{o}^{i-1^T}\right] = I$

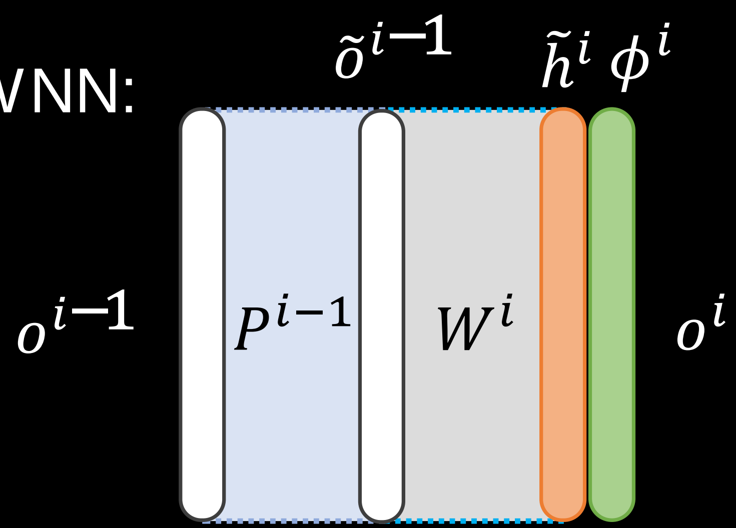
(whitening)

$\tilde{h}^{i} = W^{i}\tilde{o}^{i-1}$

(linear transformation)

Whitened Neural Networks (WNN)

WNN:



Forward: $\tilde{o}^{i-1} = P^{i-1}(o^{i-1} - \mu^{i-1}), \mathrm{E}\left[\tilde{o}^{i-1}{\tilde{o}^{i-1}}^{T}\right] = I$ (whitening)

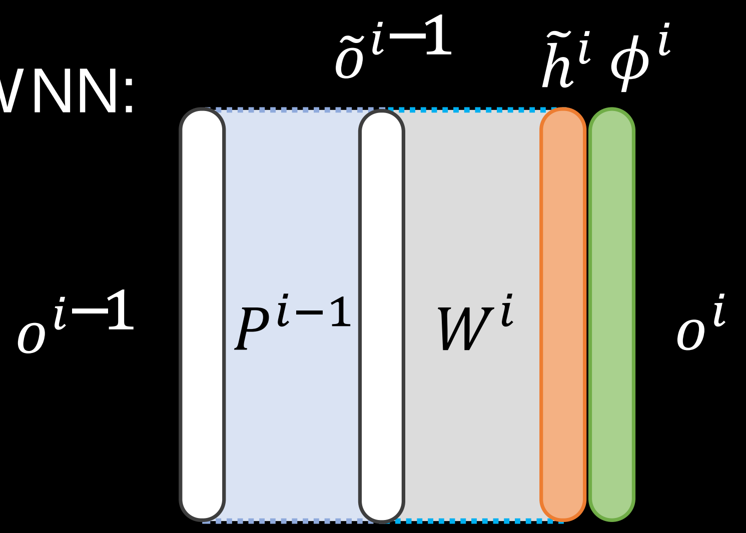
$$\tilde{h}^{i} = W^{i}\tilde{o}^{i-1}$$ (linear transformation)

$$\phi^{i} = \frac{\tilde{h}^{i}}{\sqrt{\mathrm{Var}(\tilde{h}^{i})}}$$ (batch normalization)

(nonlinear activation)

Whitened Neural Networks (WNN)

WNN:



Forward: $\tilde{o}^{i-1} = P^{i-1}(o^{i-1} - \mu^{i-1}), \mathrm{E}\left[\tilde{o}^{i-1}\tilde{o}^{i-1^T}\right] = I$ (whitening)

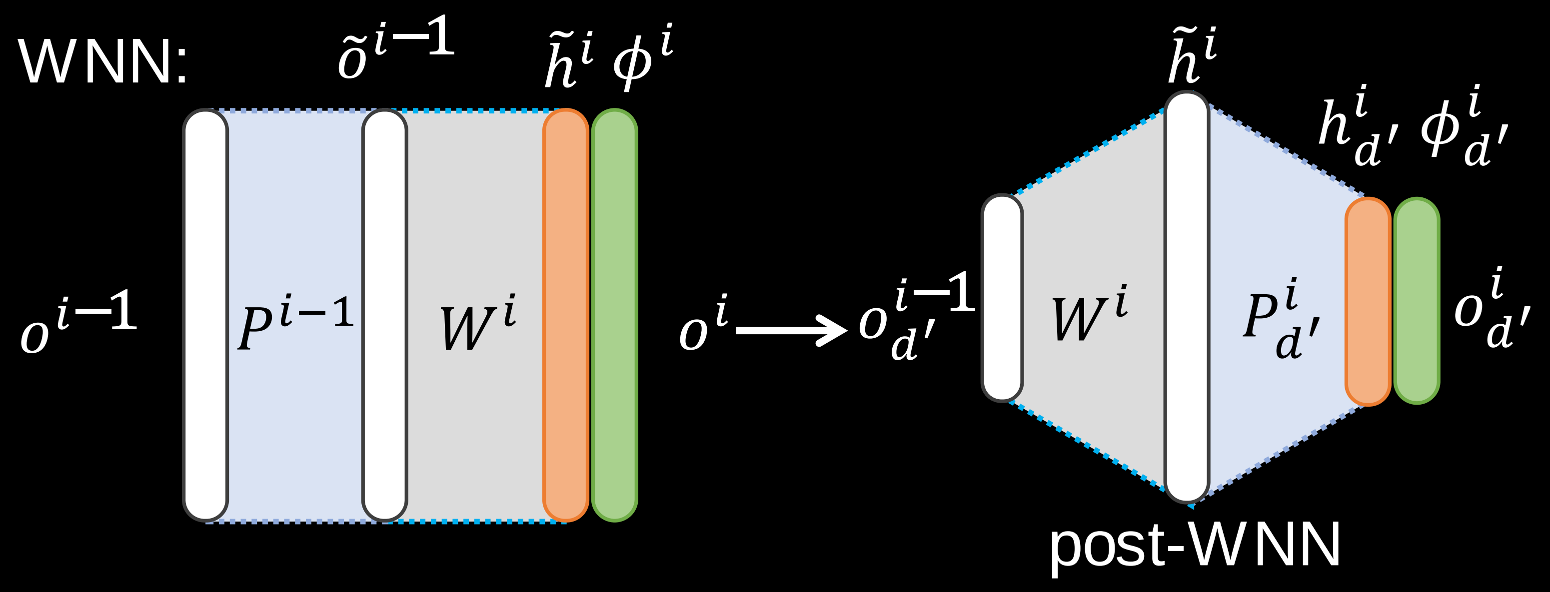
$\tilde{h}^i = W^i \tilde{o}^{i-1}$ (linear transformation)

$\phi^i = \frac{\tilde{h}^i}{\sqrt{\mathrm{Var}(\tilde{h}^i)}}$ (batch normalization)

(nonlinear activation)

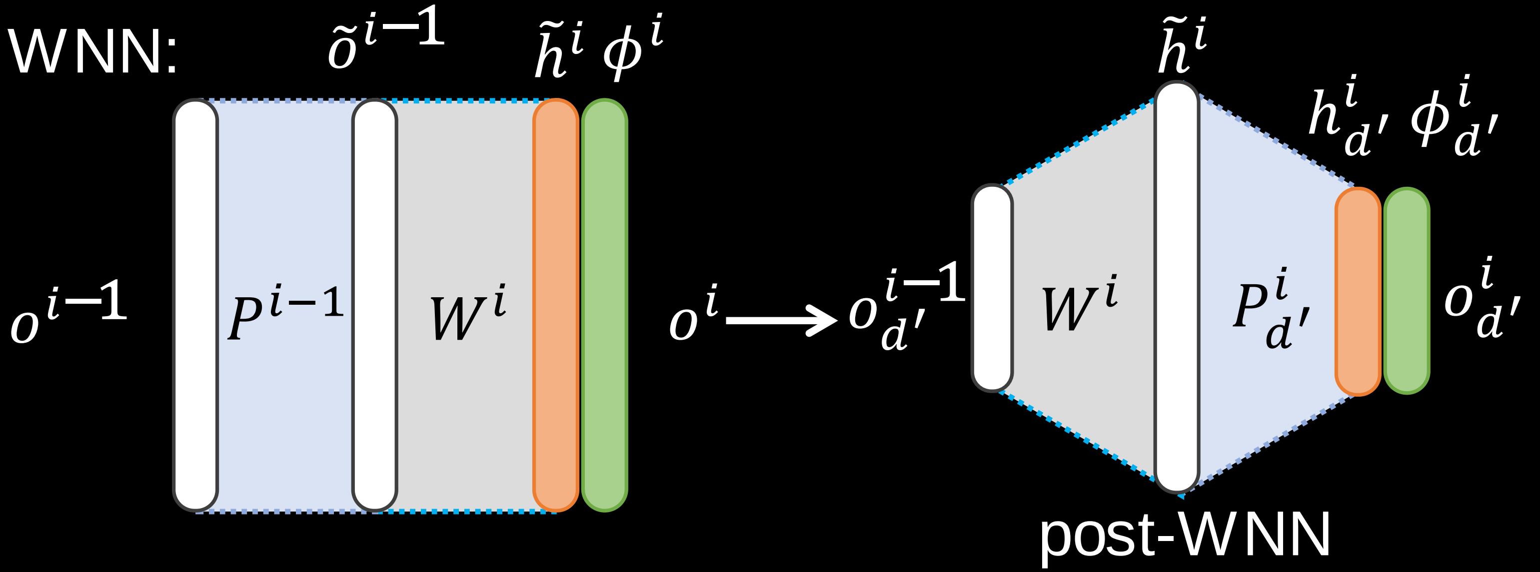
- In WNN *(Desjardins etal. NIPS15)* and GWNN *(Luo ICML17)*, $E\left[\tilde{o}^{i-1}\tilde{o}^{i-1^T}\right] = I$.
- In EigenNet *(Luo IJCAI17)*, each diagonal block of Fisher matrix F, $F_{ii} = E[\delta\tilde{h}^i \, \delta\tilde{h}^{i^T} \otimes \tilde{o}^{i-1}\tilde{o}^{i-1^T}] \approx I$.

From WNN to Post-WNN



Forward：

From WNN to Post-WNN



Forward：

$$\tilde{h}^i = W^i(o^{i-1}_{d'} - \mu^{i-1}_{d'})$$

(linear transformation)

$$h^i_{d'} = P^i_{d'}\tilde{h}^i, \qquad \mathrm{E}\left[h^i_{d'} {h^i_{d'}}^T\right] = I$$

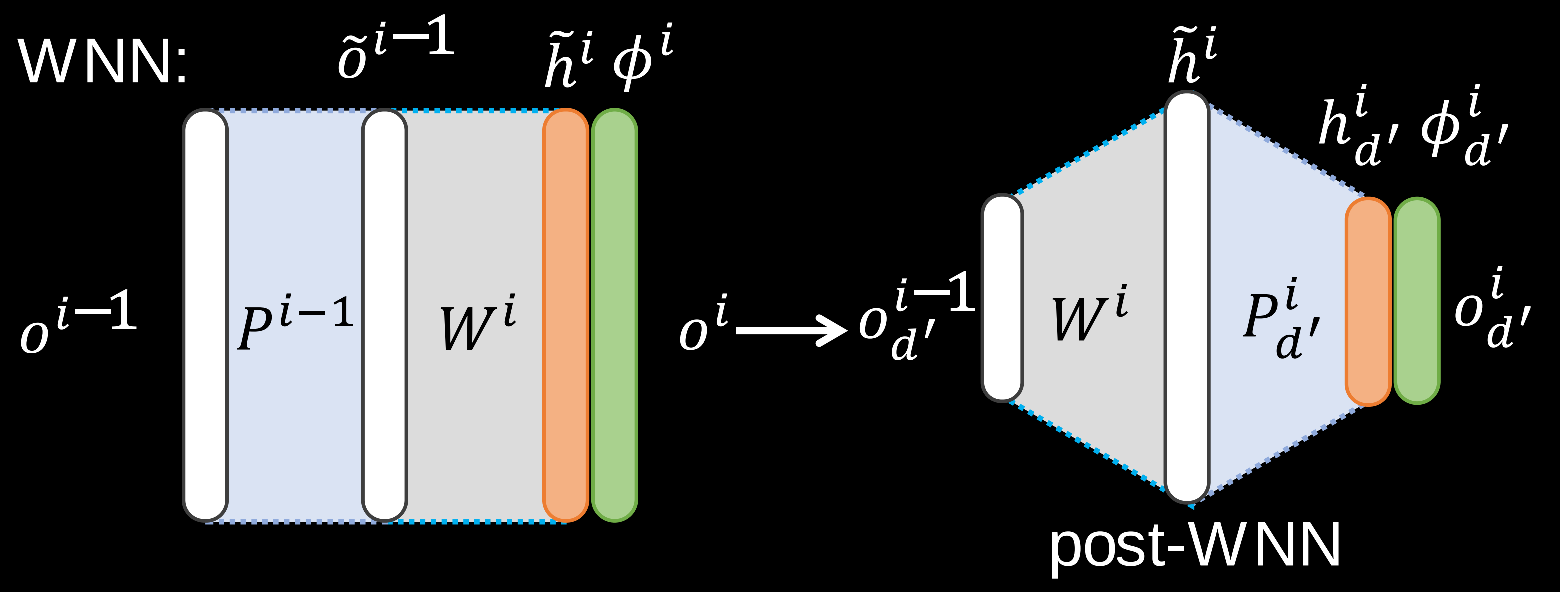
(truncated whitening)

$$\phi^i_{d'} = \frac{h^i_{d'}}{\sqrt{\mathrm{Var}(h^i_{d'})}}$$

(batch normalization)

(nonlinear activation)

From WNN to Post-WNN



Forward：

$$\tilde{h}^i = W^i(o_{d'}^{i-1} - \mu_{d'}^{i-1})$$

(linear transformation)

$$h_{d'}^i = P_{d'}^i \tilde{h}^i, \qquad \mathrm{E}\left[h_{d'}^i {h_{d'}^i}^T\right] = I$$

(truncated whitening)

$$\phi_{d'}^i = \frac{h_{d'}^i}{\sqrt{\mathrm{Var}(h_{d'}^i)}}$$

(batch normalization)

(nonlinear activation)

Post-whitening the i-th layer smooth solution space of W^{i+1}.

Summary

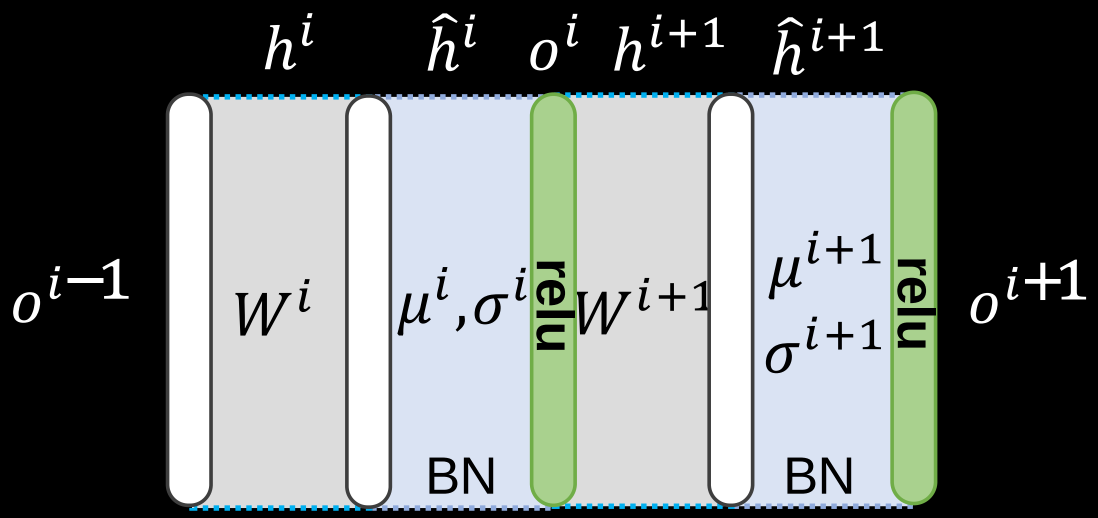
WNN, GWNN, and EigenNet are NGD.

Deep Learning turns Optimization Problems into Feed-Forward Network Design.

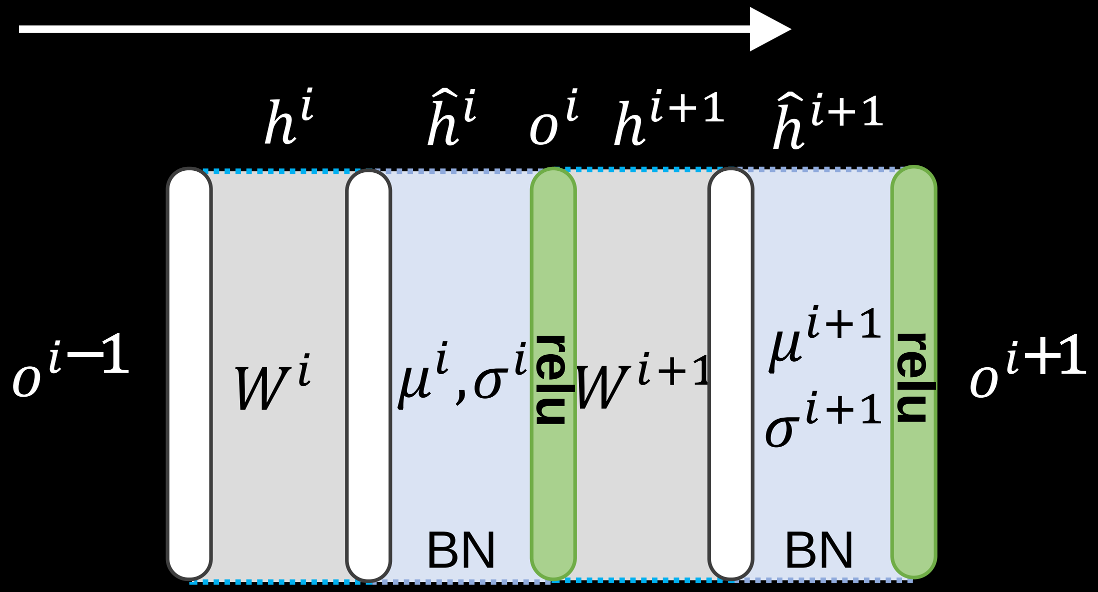
Batch Normalization (BN)

PART 2

Understand BN



Understand BN



Activations of single neuron

BN: $\hat{h}^i = \gamma^i \frac{h^i - \mathbb{E}[h^i]}{\sqrt{\mathrm{Var}(h^i)}} + \beta^i$, where

$h^i = [f_1, f_2, \ldots, f_M]$, M is minibatch size.

Understand BN

$\hat{h}^{i^T}\mathbf{1} = 0,\ \hat{h}^{i^T}\hat{h}^i = M({\gamma^i}^2 + {\beta^i}^2)$

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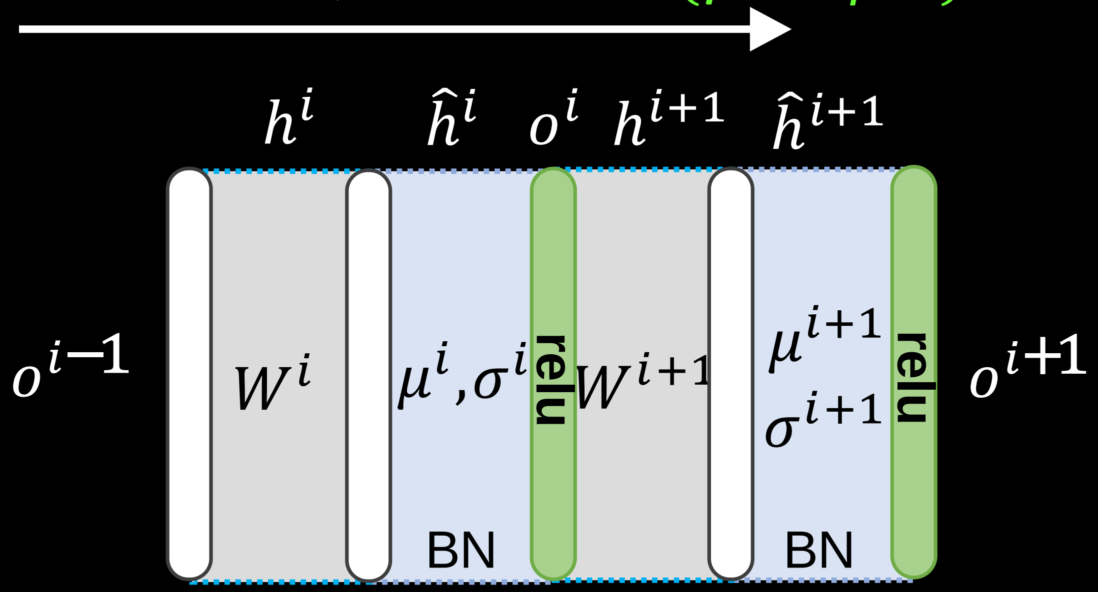

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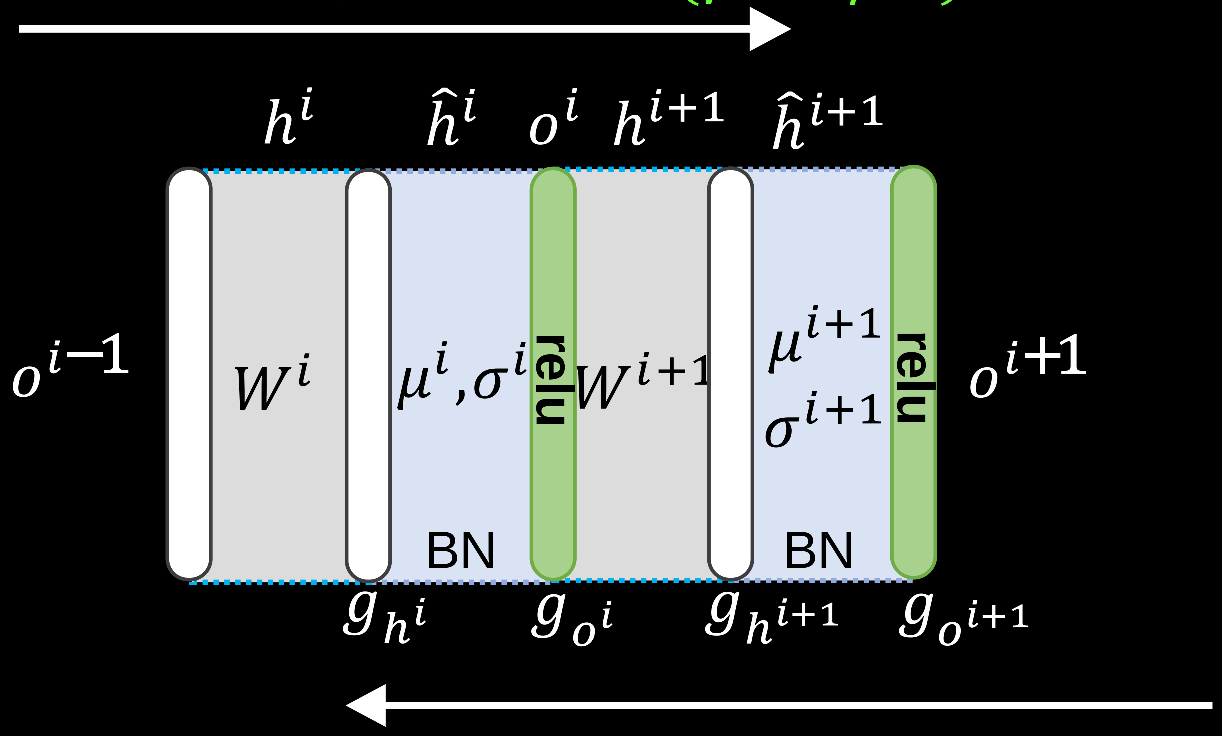

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Understand BN

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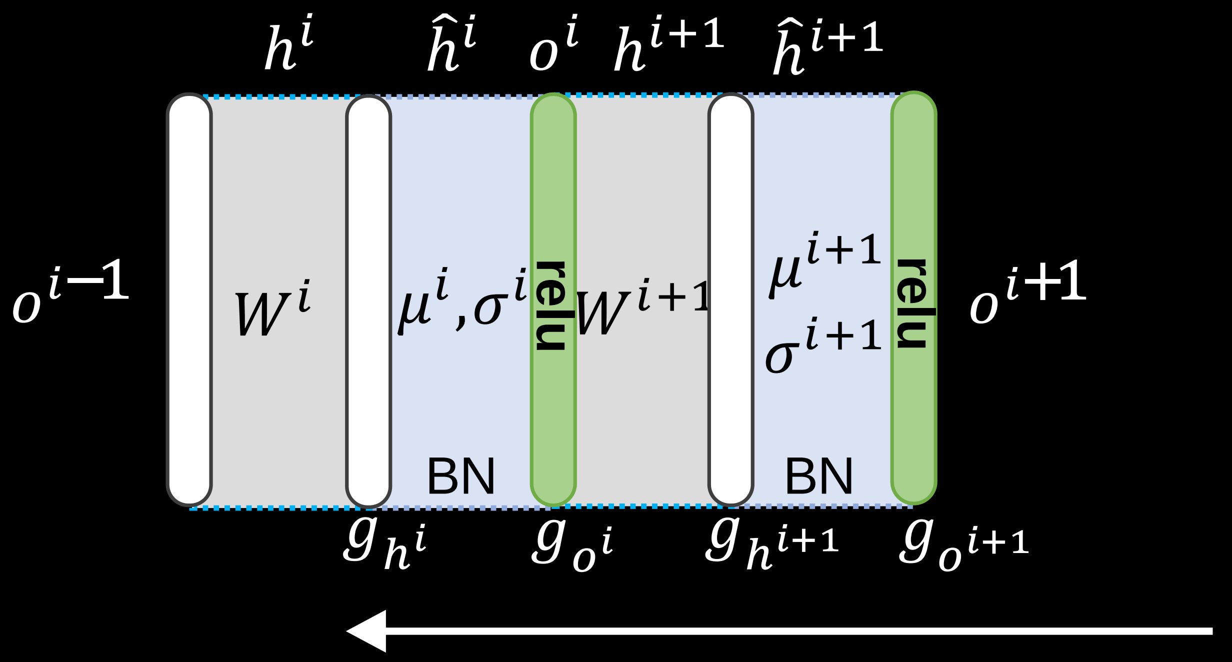
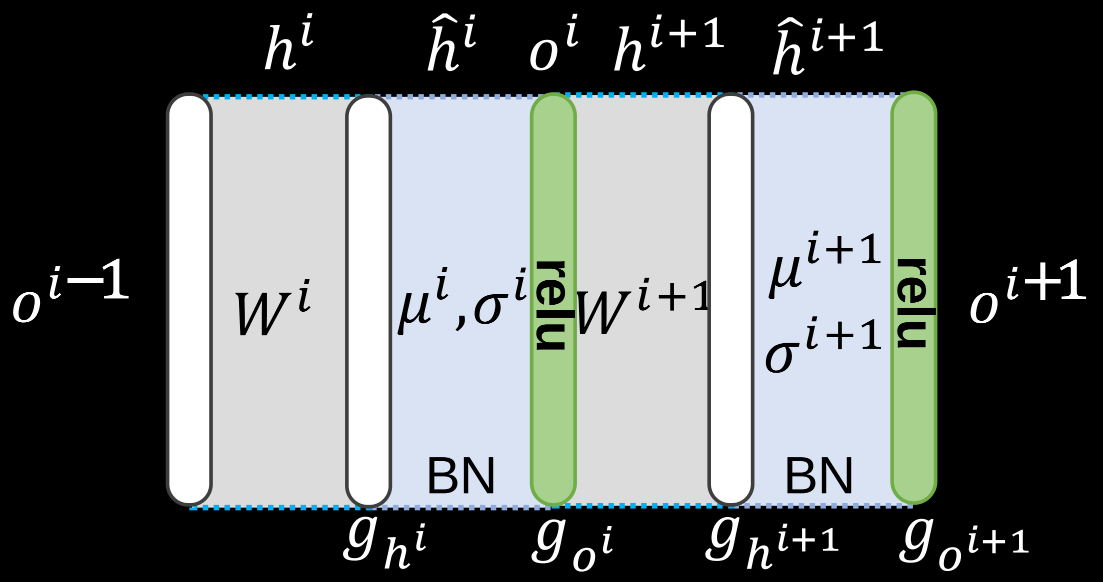


$$g_{h^{i+1}} = J^{BN} g_{o^{i+1}},\ \ J^{BN} = \frac{\partial \hat{h}^{i+1}}{\partial h^{i+1}} = \frac{\gamma^{i+1}}{\sigma^{i+1}}\left(I - \frac{[\tilde{h}^{i+1}, \mathbf{1}][\tilde{h}^{i+1}, \mathbf{1}]^T}{M}\right)$$

Understand BN

$$\hat{h}^{i^T}\mathbf{1} = 0,\ \ \hat{h}^{i^T}\hat{h}^{i} = M(\gamma^{i^2} + \beta^{i^2})$$



BN: $\hat{h}^i = \gamma^i \frac{h^i - \mathbb{E}[h^i]}{\sqrt{\mathrm{Var}(h^i)}} + \beta^i$, where

$h^i = [f_1, f_2, \ldots, f_M]$, M is minibatch size.

$$\mathrm{tr}(J^{BN}) = \frac{\gamma^{i+1}}{\sigma^{i+1}}(M-2),$$

$$\sigma^{i+1} = f(\gamma^i, \beta^i, W^i).$$

$$g_{h^{i+1}} = J^{BN} g_{o^{i+1}},\ \ J^{BN} = \frac{\partial \hat{h}^{i+1}}{\partial h^{i+1}} = \frac{\gamma^{i+1}}{\sigma^{i+1}}\left(I - \frac{[\tilde{h}^{i+1}, \mathbf{1}][\tilde{h}^{i+1}, \mathbf{1}]^T}{M}\right)$$

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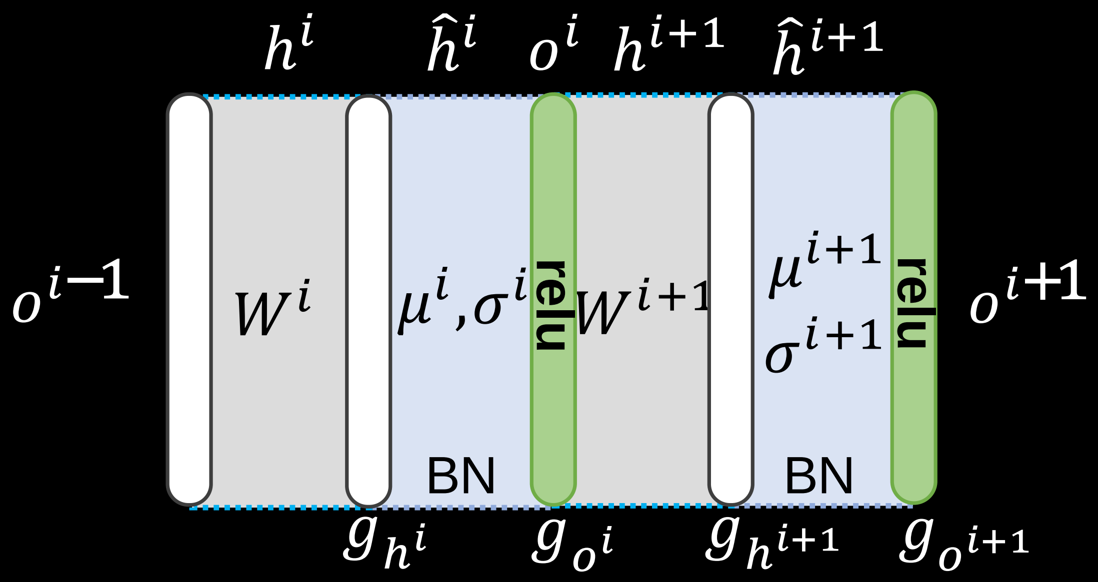

BN: $\hat{h}^i = \gamma^i \frac{h^i - \mathbb{E}[h^i]}{\sqrt{\mathrm{Var}(h^i)}} + \beta^i$, where

$h^i = [f_1, f_2, \dots, f_M]$, M is minibatch size.

$$F^{(i+1,i+1)} = \mathbb{E}[g_{h^{i+1}} g_{h^{i+1}}^T] \otimes \mathbb{E}[o^i {o^i}^T],$$

$$o^i = \max(0, \hat{h}^i)$$

$$g_{h^{i+1}} = J^{BN} g_{o^{i+1}},\quad J^{BN} = \frac{\partial \hat{h}^{i+1}}{\partial h^{i+1}} = \frac{\gamma^{i+1}}{\sigma^{i+1}}\left(I - \frac{[\tilde{h}^{i+1}, \mathbf{1}][\tilde{h}^{i+1}, \mathbf{1}]^T}{M}\right)$$

Understand BN

BN Preserves Forward and Backward Information Flows Depended on γ.

Forward of i*-th BN and Backward of* $(i+1)$*-th BN Smooth the* $(i+1)$*-th ConvLayer.*

Regularization in BN

BN: $\hat{h}_j = \gamma \frac{h_j - \mathbb{E}[h]}{\sqrt{\mathrm{Var}(h)}} + \beta$, where $\boldsymbol{h} = [h_1, h_2, \ldots h_j, \ldots, h_M]$

Regularization in BN

BN: $\hat{h}_j = \gamma \frac{h_j - \mathbb{E}[h]}{\sqrt{\mathrm{Var}(h)}} + \beta$, where $\boldsymbol{h} = [h_1, h_2, \dots h_j, \dots, h_M]$

BN is an implicit Regularizer.

Regularization in BN

sample: x^j

BN: $\hat{h}_j = \gamma \frac{h_j - \mathbb{E}[h]}{\sqrt{\mathrm{Var}(h)}} + \beta$, where $\boldsymbol{h} = [h_1, h_2, \dots h_j, \dots, h_M]$

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BN: $\hat{h}_j = \gamma \frac{h_j - \mathbb{E}[h]}{\sqrt{\mathrm{Var}(h)}} + \beta$, where $\boldsymbol{h} = [h_1, h_2, \dots h_j, \dots, h_M]$

$\mu_{\mathcal{B}} = \mathbb{E}[h_j],\ \sigma_{\mathcal{B}} = \sqrt{\mathrm{Var}(h_j)}$

Regularization in BN

BN: $\hat{h}_j = \gamma \frac{h_j - \mu_{\mathcal{B}}}{\sigma_{\mathcal{B}}} + \beta$

$$\mu_{\mathcal{B}} \backsim \mathcal{N}(\mu_{\mathcal{P}}, \frac{\sigma_{\mathcal{P}}^2}{M}), \ \sigma_{\mathcal{B}} \backsim \mathcal{N}(\sigma_{\mathcal{P}}, \frac{\rho + 2}{4M})$$

Regularization in BN

BN: $\hat{h}_j = \gamma \frac{h_j - \mu_{\mathcal{B}}}{\sigma_{\mathcal{B}}} + \beta$

$$\mu_{\mathcal{B}} \backsim \mathcal{N}(\mu_{\mathcal{P}}, \frac{\sigma_{\mathcal{P}}^2}{M}),\ \ \sigma_{\mathcal{B}} \backsim \mathcal{N}(\sigma_{\mathcal{P}}, \frac{\rho+2}{4M})$$

When $M \to P$, $\mu_{\mathcal{B}} \to \mu_{\mathcal{P}}$ and $\sigma_{\mathcal{B}} \to \sigma_{\mathcal{P}}$.

Regularization in BN

$$\mathcal{BN} \approx \mathcal{PN} + \mathcal{G}amma\ \mathcal{D}ecay$$

Regularization in BN

BN: $\hat{h}_j = \gamma \frac{h_j - \mu_{\mathcal{B}}}{\sigma_{\mathcal{B}}} + \beta$

Population Normalization (PN): $\bar{h}_j = \gamma \frac{h_j - \mu_{\mathcal{P}}}{\sigma_{\mathcal{P}}} + \beta$

Regularization in BN

BN: $\hat{h}_j = \gamma \frac{h_j - \mu_{\mathcal{B}}}{\sigma_{\mathcal{B}}} + \beta$ PN: $\bar{h}_j = \gamma \frac{h_j - \mu_{\mathcal{P}}}{\sigma_{\mathcal{P}}} + \beta$

An explicit regularizer:

Regularization in BN

BN: $\hat{h}_j = \gamma \frac{h_j - \mu_{\mathcal{B}}}{\sigma_{\mathcal{B}}} + \beta$ PN: $\bar{h}_j = \gamma \frac{h_j - \mu_{\mathcal{P}}}{\sigma_{\mathcal{P}}} + \beta$

An explicit regularizer:

$$\frac{1}{P}\sum_{j=1}^{P} \mathbb{E}_{\mu_{\mathcal{B}},\sigma_{\mathcal{B}}}[\mathcal{L}(\hat{h}_j^i)] \approx \frac{1}{P}\sum_{j=1}^{P} \mathcal{L}\left(\bar{h}_j^i\right) + \zeta\left(h_j^i\right)\gamma^2,$$

and $$\zeta\left(h_j^i\right) = \frac{\rho + 2}{8M} F_\gamma + \frac{1}{2M}\frac{1}{P}\sum_{j=1}^{P} \sigma(\bar{h}_j^i).$$

Regularization in BN

BN: $\hat{h}_j = \gamma \frac{h_j - \mu_{\mathcal{B}}}{\sigma_{\mathcal{B}}} + \beta$ PN: $\bar{h}_j = \gamma \frac{h_j - \mu_{\mathcal{P}}}{\sigma_{\mathcal{P}}} + \beta$

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and $$\zeta(h_j^i) = \underbrace{\frac{\rho+2}{8M}F_\gamma}_{\text{from } \sigma_{\mathcal{B}}} + \underbrace{\frac{1}{2M}\frac{1}{P}\sum_{j=1}^{P}\sigma(\bar{h}_j^i)}_{\text{from } \mu_{\mathcal{B}}}.$$

Regularization in BN

$$\zeta(h_j^i) = \underbrace{\frac{\rho + 2}{8M}}_{\text{from } \sigma_{\mathcal{B}}} F_\gamma + \underbrace{\frac{1}{2M}\frac{1}{P}\sum_{j=1}^{P}\sigma(\bar{h}_j^i)}_{\text{from } \mu_{\mathcal{B}}}.$$

Regularization in BN

punish gradient norm and correlations.

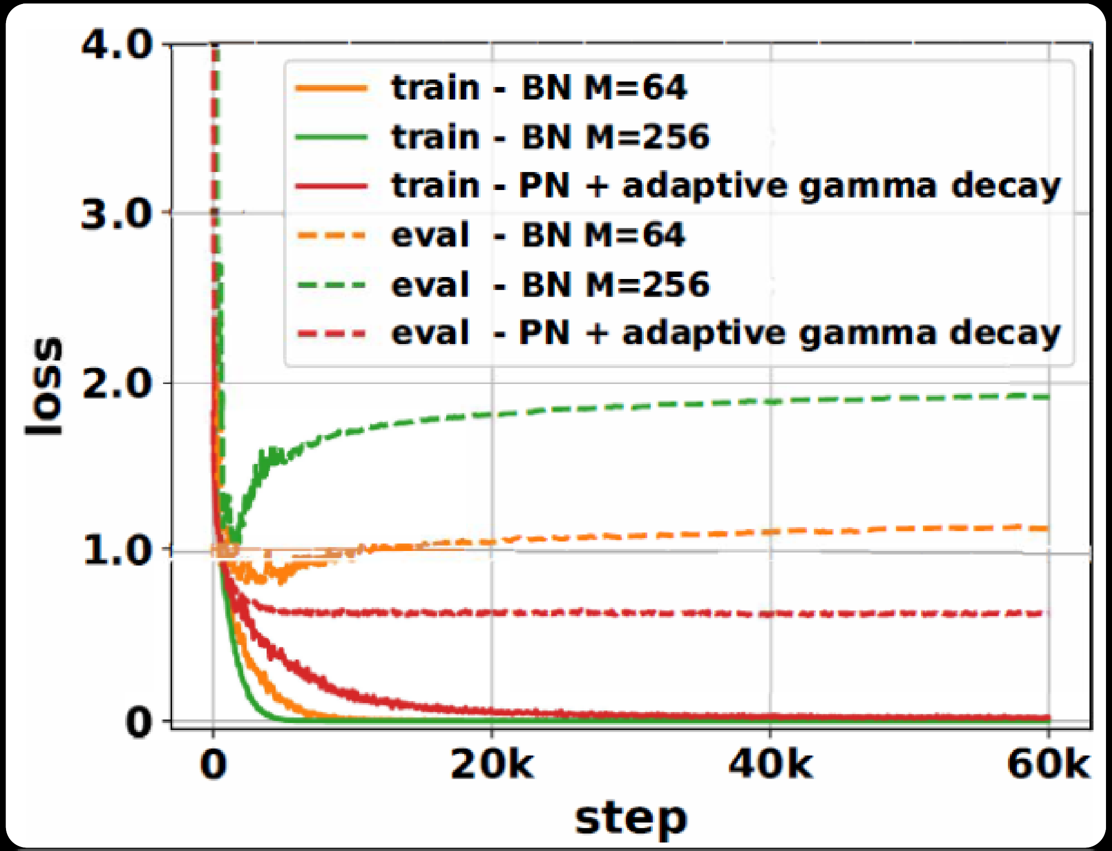
prevent reliance on single neuron.

$$\zeta(h_j^i) = \underbrace{\frac{\rho + 2}{8M} F_\gamma}_{\text{from } \sigma_{\mathcal{B}}} + \underbrace{\frac{1}{2M} \frac{1}{P} \sum_{j=1}^{P} \sigma(\bar{h}_j^i)}_{\text{from } \mu_{\mathcal{B}}}.$$

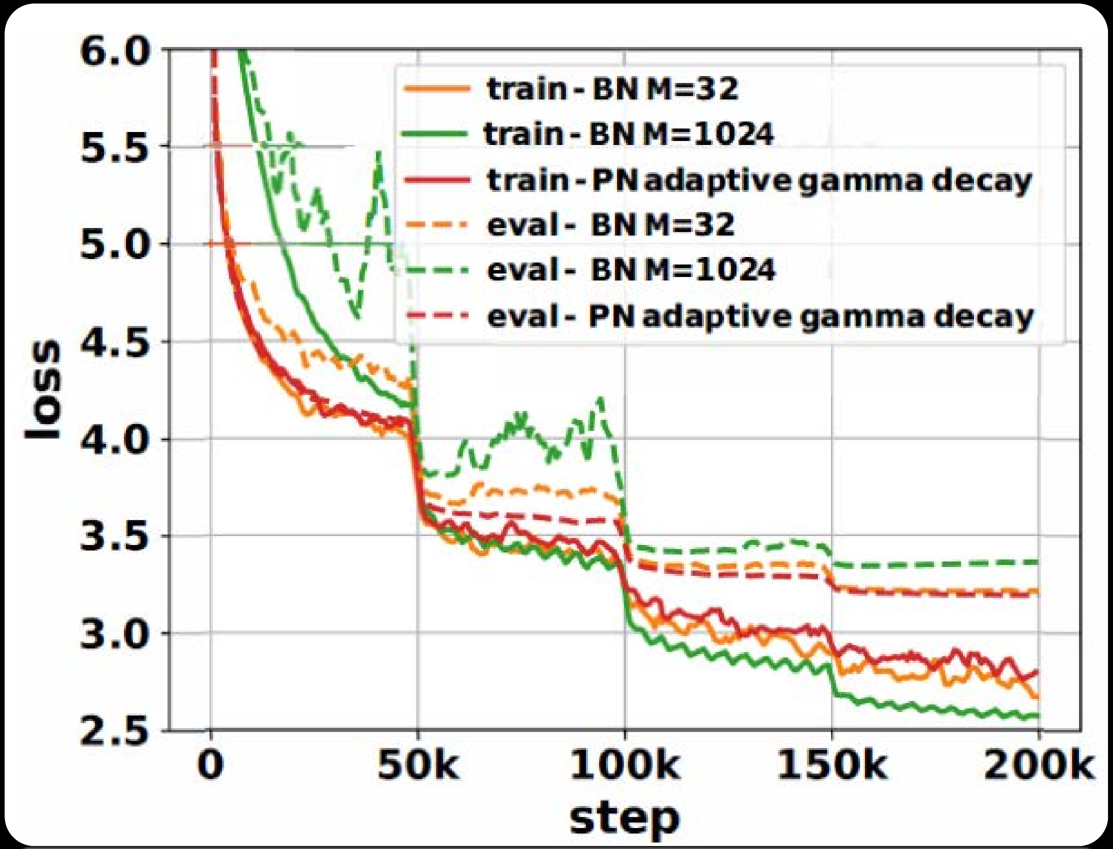
Regularization in BN

μ and σ in BN have Different Regularization Impacts.

Regularization in BN

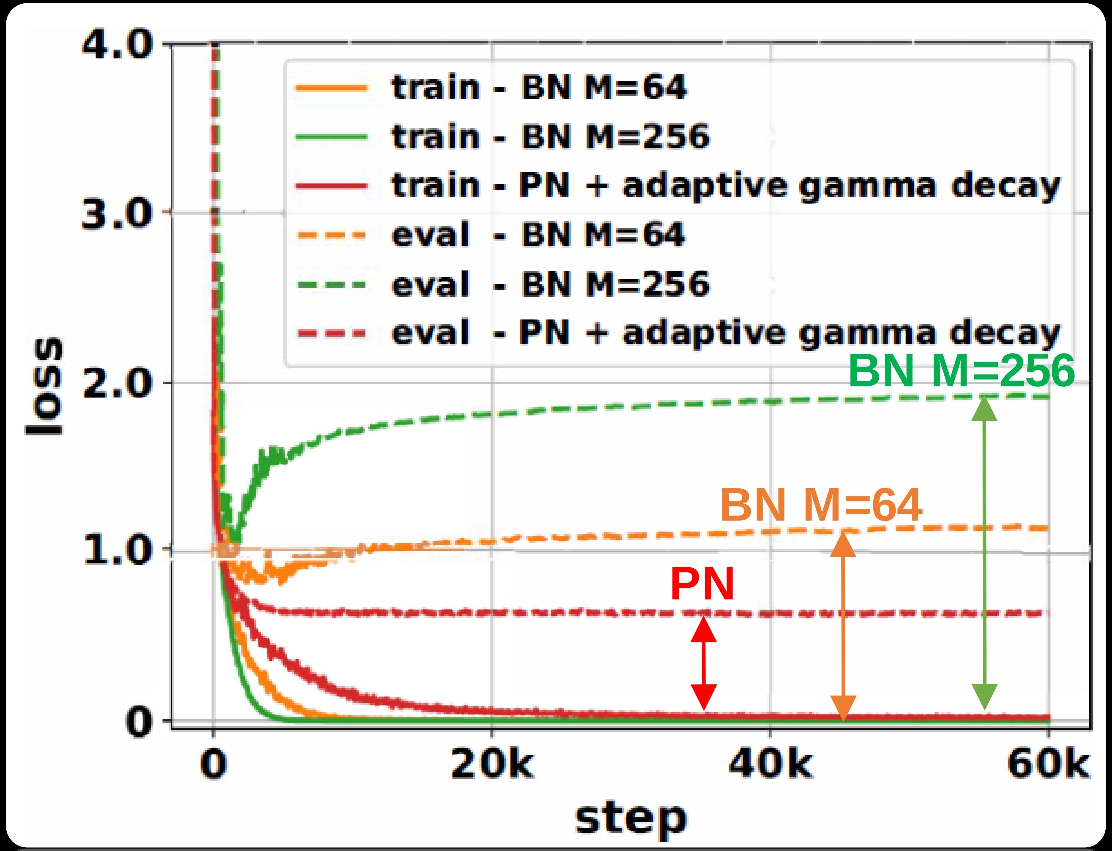


CIFAR10 (CNN6)



down-sampled ImageNet (R18)

Regularization in BN



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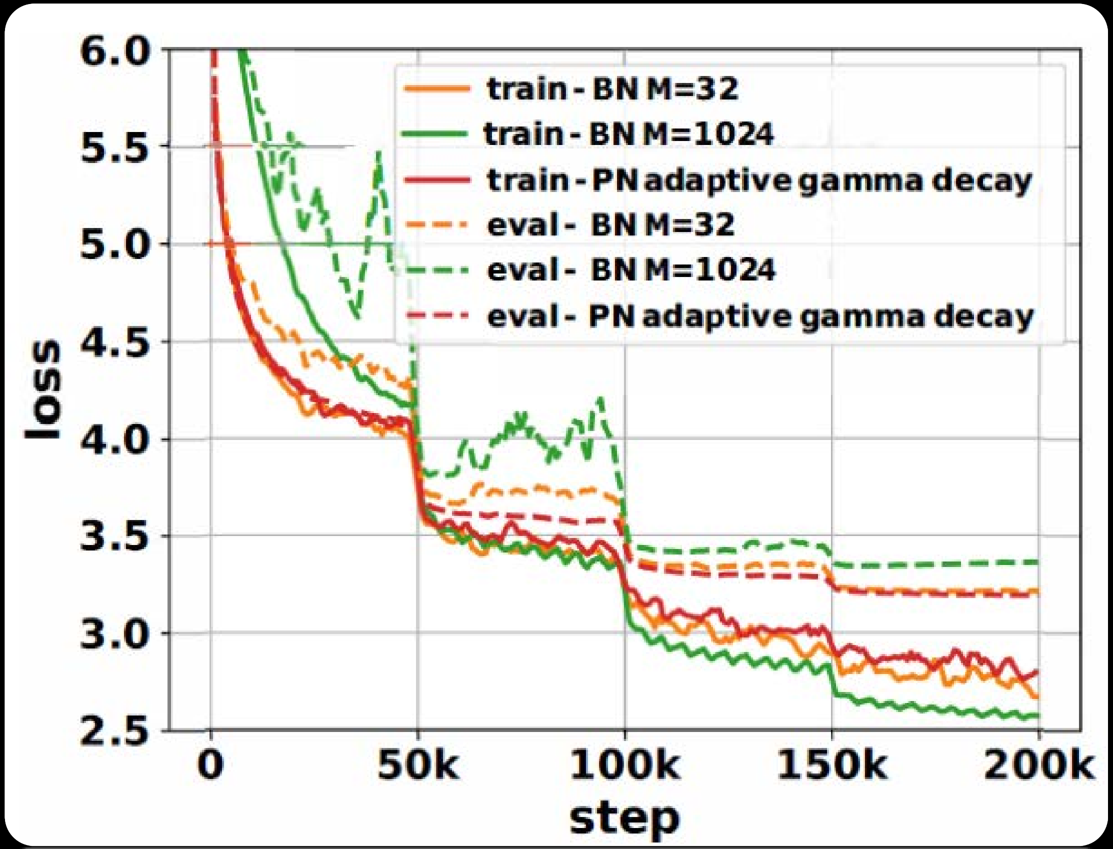


down-sampled ImageNet (R18)

Summary

BN Preserves Forward and Backward Information Flows Depended on γ.

Forward of i*-th BN and Backward of* $(i+1)$*-th BN Smooth the* $(i+1)$*-th ConvLayer.*

Summary

BN is an implicit Regularizer.

BN ≈ PN + Gamma Decay

μ and σ in BN have Different Regularization Impacts.

BN turns data-dependent Regularizations into Feed-forward Estimation of Batch Statistics.

Switchable Normalization (SN)

PART 3

Switchable Normalization (SN)

A new Perspective for Deep Learning:

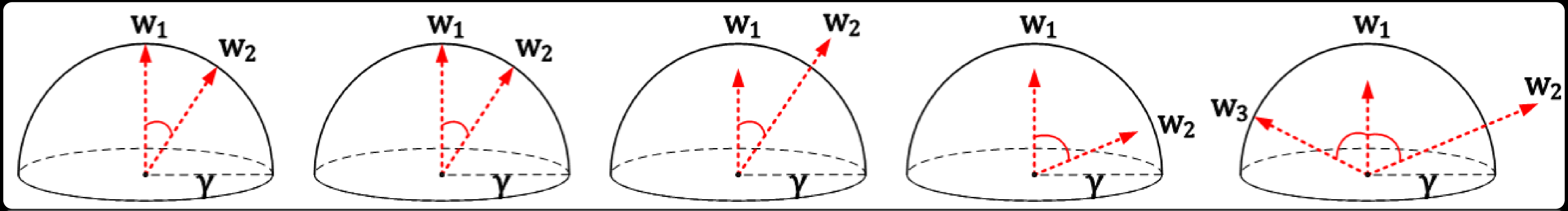
Each ConvLayer in a ConvNet needs its own Normalizer.

Switchable Normalization (SN)

$$\hat{h}_{\mathrm{SN}} = \frac{h - \sum_{z\in\Omega} \lambda_z \mu_z}{\sum_{z\in\Omega} \lambda'_z \sigma_z}, \quad \Omega = \{\mathrm{IN}, \mathrm{LN}, \mathrm{BN}\},$$

$$\forall z,\ \sum_{z\in\Omega} \lambda_z = 1,\ \sum_{z\in\Omega} \lambda'_z = 1.$$

Switchable Normalization (SN)



WN: $\frac{\boldsymbol{w}_c^T \boldsymbol{x}}{\|\boldsymbol{w}_c\|}$

IN: $\frac{\boldsymbol{w}_c^T \boldsymbol{x}}{\|\boldsymbol{w}_c\|}$

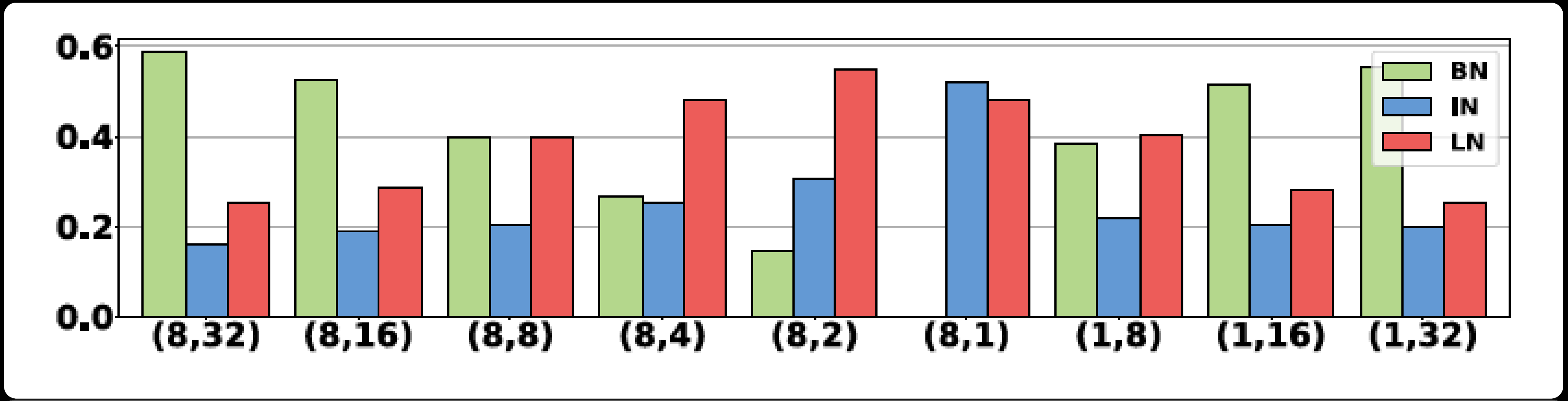
LN: $\frac{\boldsymbol{w}_c^T \boldsymbol{x}}{\sum_{c=1}^{C} \|\boldsymbol{w}_c\|}$

BN: $\frac{\boldsymbol{w}_c^T \boldsymbol{x}}{\|\boldsymbol{w}_c\|}$

reg. $\zeta(h)\gamma^2$

SN: $\hat{h}_{\mathrm{SN}} = \lambda_{\mathrm{BN}} \hat{h}_{\mathrm{BN}} + \lambda_{\mathrm{IN}} \hat{h}_{\mathrm{IN}} + \lambda_{\mathrm{LN}} \hat{h}_{\mathrm{LN}}$

Switchable Normalization (SN)



When batch size decreases, LN ratio increases while BN decreases.

Switchable Normalization (SN)

backbone	head	AP	$AP_{.5}$	$AP_{.75}$	AP_l	AP_m	AP_s
$BN^\dagger$	–	36.7	58.4	39.6	48.1	39.8	21.1
$BN^\dagger$	GN	37.2	58.0	40.4	48.6	40.3	21.6
$BN^\dagger$	SN	38.0	59.4	41.5	48.9	41.3	22.7
GN	GN	38.2	58.7	41.3	49.6	41.0	22.4
SN	SN	**39.3**	**60.9**	**42.8**	**50.3**	**42.7**	**23.5**

Table 3: **Faster R-CNN+FPN** using ResNet50 and FPN with 1x LR schedule. $BN^\dagger$ represents BN is frozen. The best results are bold.

backbone	head	AP^b	$AP^b_{.5}$	$AP^b_{.75}$	AP^m	$AP^m_{.5}$	$AP^m_{.75}$
$BN^\dagger$	–	38.6	59.5	41.9	34.2	56.2	36.1
$BN^\dagger$	GN	39.5	60.0	43.2	34.4	56.4	36.3
$BN^\dagger$	SN	40.0	61.0	43.3	34.8	57.3	36.3
GN	GN	40.2	60.9	43.8	35.7	57.8	38.0
GN	SN	40.4	61.4	44.2	36.0	58.4	38.1
SN	SN	**41.0**	**62.3**	**45.1**	**36.5**	**58.9**	**38.7**

Table 4: **Mask R-CNN** using ResNet50 and FPN with 2x LR schedule. $BN^\dagger$ represents BN is frozen without finetuning. The best results are bold.

Switchable Normalization (SN)

	ADE20K		Cityscapes	
	mIoU_{ss}	mIoU_{ms}	mIoU_{ss}	mIoU_{ms}
SyncBN	36.4	37.7	69.7	73.0
GN	35.7	36.3	68.4	73.1
SN	**38.7**	**39.2**	**71.2**	**75.1**

Table 5: **Results in ADE20K validation set and Cityscapes test set** by using ResNet50 with dilated convolutions. ‘ss’ and ‘ms’ indicate single-scale and multi-scale inference. SyncBN represents mutli-GPU synchronization of BN. SN finetunes from $(8, 2)$ pretrained model.

Switchable Normalization (SN)

	batch=8, length=32		batch=4, length=32	
	top1	top5	top1	top5
BN	73.2	90.9	72.1	90.0
GN	73.0	90.6	72.8	90.6
SN	**73.5**	**91.3**	**73.3**	**91.2**

Table 6: **Results of Kinetics dataset.** In training, the clip length of 32 frames is regularly sampled with a frame interval of 2. We study a batch size of 8 or 4 clips per GPU. BN is not synchronized across GPUs. SN finetunes from $(8, 2)$ pretrained model.

Summary

A Novel Viewpoint for Deep Learning:

Different Normalization Layers in a Deep ConvNet

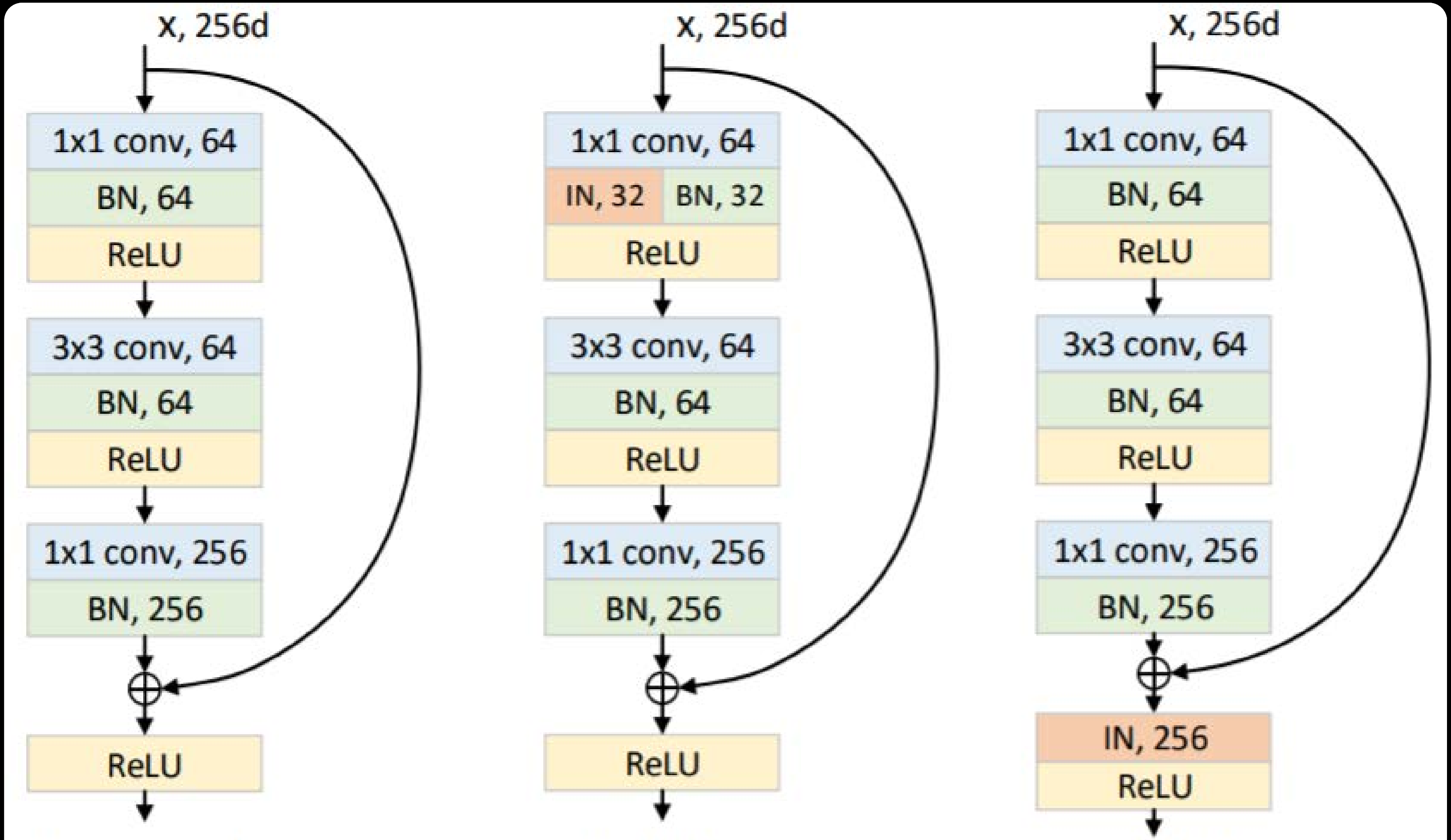
Require Different Normalizers.

SN would be valuable in “any” problem that needs normalizations, according to its theoretical property.

Instance-Batch Normalization (IBN-Net) and Kalman Normalization (KN)

PART 4

Instance-Batch Normalization



(a) original (b) IBN-a (c) IBN-b

Instance-Batch Normalization

Table 2. Results of IBN-Net over other CNNs on ImageNet validation set. The performance gains are shown in the brackets. More detailed descriptions of these IBN-Nets are provided in the supplementary material.

Model	original	re-implementation	IBN-Net-a
	top1/top5 err.	top1/top5 err.	top1/top5 err.
DenseNet121 [13]	25.0/-	24.96/7.85	24.47/7.25 (0.49/0.60)
DenseNet169 [13]	23.6/-	24.02/7.06	23.25/6.51 (0.79/0.55)
ResNet50 [8]	24.7/7.8	24.27/7.08	22.54/6.32 (1.73/0.76)
ResNet101 [8]	23.6/7.1	22.48/6.23	21.39/5.59 (1.09/0.64)
ResNeXt101 [31]	21.2/5.6	21.31/5.74	20.88/5.42 (0.43/0.32)
SE-ResNet101 [12]	22.38/6.07	21.68/5.88	21.25/5.51 (0.43/0.37)

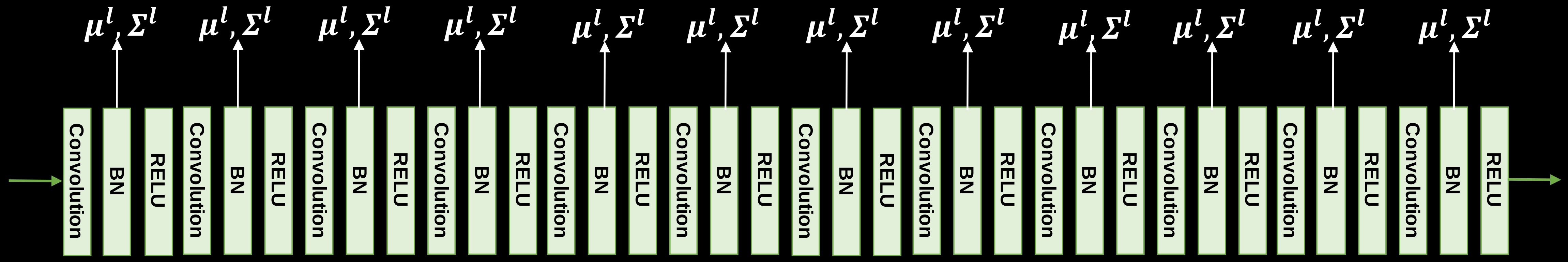
Instance-Batch Normalization

Table 6. Results on Cityscapes-GTA dataset. Mean IoU for both within domain evaluation and cross domain evaluation is reported.

Train	Test	Model	mIoU(%)	Pixel Acc.(%)
Cityscapes	Cityscapes	ResNet50	64.5	93.4
		IBN-Net50-a	**69.1**	**94.4**
		IBN-Net50-b	67.0	94.3
	GTA5	ResNet50	29.4	71.9
		IBN-Net50-a	32.5	71.4
		IBN-Net50-b	**37.9**	**78.8**
GTA5	GTA5	ResNet50	61.0	91.5
		IBN-Net50-a	**64.8**	**92.5**
		IBN-Net50-b	64.2	92.4
	Cityscapes	ResNet50	22.2	53.5
		IBN-Net50-a	26.0	60.9
		IBN-Net50-b	**29.6**	**66.8**

Kalman Normalization (KN)

- Propagate statistics in BN through the entire CNN by learning a transition matrix E.
- KN is designed for training with micro-batch.



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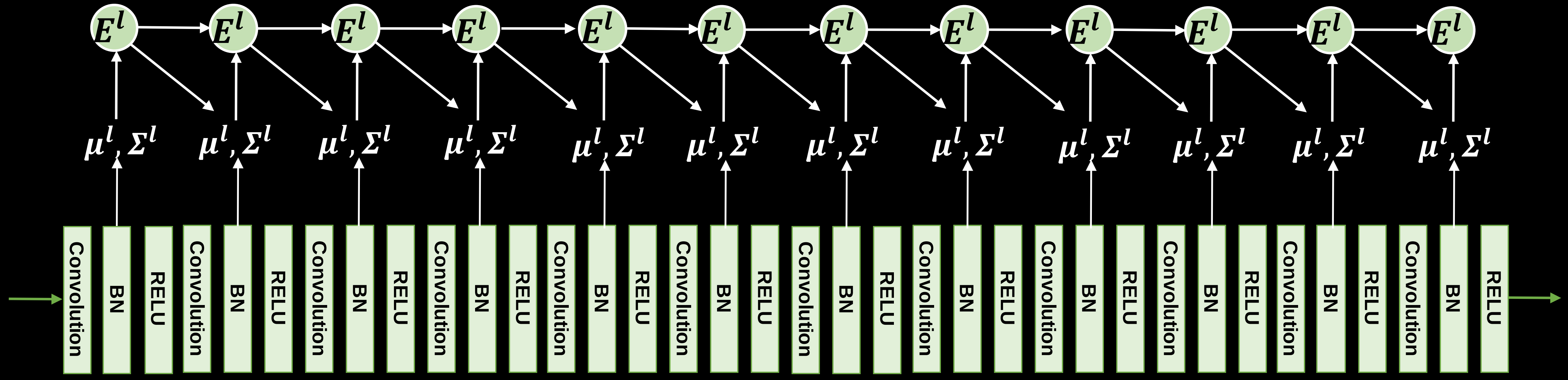

Comparisons

ResNet50	BN	GN	SN	KN
256, 32	76.4	75.9	**77.2**	76.8
32, 4	72.7	75.8	**75.9**	-
256, 4	-	-	**77.2**	76.1

Discussions and Future Work

PART 5

Recap of Results

1. Deep Learning turns optimization problem into feed-forward computations, e.g. WNN, GWNN, and EigenNet are NGD.

Recap of Results

1. Deep Learning turns optimization problem into feed-forward computations, e.g. WNN, GWNN, and EigenNet are NGD.
2. BN is an implicit regularizer, whose explicit form is “BN ≈ PN + Gamma Decay”.
   - Compute regularization in batch to replace batch statistics in BN.

Recap of Results

1. Deep Learning turns optimization problem into feed-forward computations, e.g. WNN, GWNN, and EigenNet are NGD.
2. BN is an implicit regularizer, whose explicit form is “BN ≈ PN + Gamma Decay”.
   - Compute regularization in batch to replace batch statistics in BN.
3. New research direction: *different normalization layers in a ConvNet use different normalizers.*
   - Switchable Normalization (SN) would be applicable in “any” problem because of its characteristic.

Working Papers

- ✓ “Do Normalization Layers in a Deep ConvNet Really Need to be Distinct?”
- ✓ “Learning Sparse Switchable Normalization via SparsestMax”
  - SoftMax $\rightarrow$ SparseMax $\rightarrow$ SparsestMax
- ✓ Understanding Learning Dynamics and Generalization of SN
  - Loss functions, input distributions, over-parameterization
- ✓ SNv2

Codebase

https://github.com/switchablenorms

Switchable-Normalization

Code for Switchable Normalization from "Differentiable Learning-to-Normalize via Switchable Normalization", https://arxiv.org/abs/1806.10779

HTML 498 71

SwitchNorm_Detection

Forked from roytseng-tw/Detectron.pytorch

The code of Switchable Normalization for object detection based on Detectron.pytorch.

Python 68 9

SwitchNorm_Segmentation

Switchable Normalization for semantic image and scene parsing.

Python 18 1

SSN, SparsestMax, SNv2

SN: Kinetics, MegaFace, GANs, …

https://github.com/XingangPan/IBN-Net

IBN-Net

Instance-Batch Normalization Networks (ECCV2018)

Python 321 49

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Reference

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Appendix

PART 6

Kalman Normalization (KN)

True value of h^l:

$$h^l = E^l h^{l-1} + u^l, \quad u^l \sim \mathcal{N}(0, R^l)$$

transition matrix   bias

Kalman Normalization (KN)

True value of h^l:

$$h^l = E^l h^{l-1} + u^l, \quad u^l \sim \mathcal{N}(0, R^l)$$

$$z^l = h^l + v^l$$

bias

observed value of a mini-batch

Kalman Normalization (KN)

True value of h^l:

$$h^l = E^l h^{l-1} + u^l, \quad u^l \sim \mathcal{N}(0, R^l)$$

$$z^l = h^l + v^l$$

Estimated mean of h^l:

$$\hat{\mu}^{l|l-1} = \mathbb{E}[h^l] = \mathbb{E}[E^l h^{l-1} + u^l]$$

Kalman Normalization (KN)

True value of h^l:

$$h^l = E^l h^{l-1} + u^l, \quad u^l \sim \mathcal{N}(0, R^l)$$

$$z^l = h^l + v^l$$

Estimated mean of h^l:

$$\hat{\mu}^{l|l-1} = \mathbb{E}[h^l] = \mathbb{E}[E^l h^{l-1} + u^l]$$

Estimated variance of h^l:

$$\hat{\Sigma}^{l|l-1} = \mathrm{Cov}(h^l - \hat{\mu}^{l|l-1}) = E^l \hat{\Sigma}^{l-1|l-1} {E^l}^T + R^l$$

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Estimated mean of h^l:

$$\hat{\mu}^{l|l-1} = \mathbb{E}[h^l] = \mathbb{E}[E^l h^{l-1} + u^l]$$

$$\hat{\mu}^{l|l} = \hat{\mu}^{l|l-1} + q^l (z^l - \hat{\mu}^{l|l-1})$$

estimation of mean
in current layer

estimation of mean
in previous layer

Estimated variance of h^l:

$$\hat{\Sigma}^{l|l-1} = \mathrm{Cov}(h^l - \hat{\mu}^{l|l-1}) = E^l \hat{\Sigma}^{l-1|l-1} {E^l}^T + R^l$$

$$\hat{\Sigma}^{l|l} = \mathrm{Cov}(h^l - \hat{\mu}^{l|l}) = \hat{\Sigma}^{l|l-1} + q^l (C^l - \hat{\Sigma}^{l|l-1}) + \left(1 - q^l\right) q^l (z^l - \hat{\mu}^{l|l-1})^2$$

Kalman Normalization (KN)

True value of h^l:

$$h^l = E^l h^{l-1} + u^l, \quad u^l \sim \mathcal{N}(0, R^l)$$

$$z^l = h^l + v^l$$

Estimated mean of h^l:

$$\hat{\mu}^{l|l-1} = \mathbb{E}[h^l] = \mathbb{E}[E^l h^{l-1} + u^l]$$

$$\hat{\mu}^{l|l} = \hat{\mu}^{l|l-1} + q^l \underbrace{(z^l - \hat{\mu}^{l|l-1})}_{\text{bias between the observed mean and intermediate estimation}}$$

estimation of mean in current layer

Estimated variance of h^l:

$$\hat{\Sigma}^{l|l-1} = \mathrm{Cov}(h^l - \hat{\mu}^{l|l-1}) = E^l \hat{\Sigma}^{l-1|l-1} {E^l}^T + R^l$$

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Estimated mean of h^l:

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estimation of mean
in current layer

gain

Estimated variance of h^l:

$$\hat{\Sigma}^{l|l-1} = \mathrm{Cov}(h^l - \hat{\mu}^{l|l-1}) = E^l \hat{\Sigma}^{l-1|l-1} {E^l}^T + R^l$$

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Estimated variance of h^l:

$$\hat{\Sigma}^{l|l-1} = \mathrm{Cov}(h^l - \hat{\mu}^{l|l-1}) = E^l \hat{\Sigma}^{l-1|l-1} {E^l}^T + R^l$$

$$\hat{\Sigma}^{l|l} = \mathrm{Cov}(h^l - \hat{\mu}^{l|l}) = \hat{\Sigma}^{l|l-1} + q^l(\boxed{C^l} - \hat{\Sigma}^{l|l-1}) + (1 - q^l)q^l(z^l - \hat{\mu}^{l|l-1})^2$$

observed covariance matrix

Kalman Normalization (KN)

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$$\hat{\mu}^{l|l-1} = \mathbb{E}[h^l] = \mathbb{E}[E^l h^{l-1} + u^l]$$

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Estimated variance of h^l:

$$\hat{\Sigma}^{l|l-1} = \mathrm{Cov}(h^l - \hat{\mu}^{l|l-1}) = \boxed{E^l} \hat{\Sigma}^{l-1|l-1} {E^l}^T + \boxed{R^l}$$

$$\hat{\Sigma}^{l|l} = \mathrm{Cov}(h^l - \hat{\mu}^{l|l}) = \hat{\Sigma}^{l|l-1} + \boxed{q^l}(C^l - \hat{\Sigma}^{l|l-1}) + (1 - q^l)q^l(z^l - \hat{\mu}^{l|l-1})^2$$

optimized in training